Linearization theorems, Koopman operator and its application

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1 Introduction
   • Linear and nonlinear dynamics
   • Local and global linearization

2 Linearization in large
   • Extensions of Hartman’s theorem
   • Examples

3 The Koopman operator
   • Its introduction
   • Koopman operator and partition of the phase space

4 Applications
   • The standard map
   • Application to fluid dynamics

5 Summary
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5. Summary
Hierarchical Structure
Many agents interconnected in a complex manner, resulting in a multi-scale, highly heterogeneous system.

Self-organized dynamics
Nonlinear, non-equilibrium dynamics leads to the emergent behavior, which is non-reducible and unpredictable from single agent behavior.

Adaptability
Robust while flexible, evolvable.

Uncertainties
The determination of parameters and relations are hard.
The internetiverse


You are here
Turbulence
Structure of macromolecules
Cell regulatory networks
Nonlinear dynamics: triumph and challenge

- **Triumph of theory of nonlinear dynamical systems**
  Local: linear stability analysis, bifurcation theory, normal form theory, ...
  Global: Asymptotic analysis, topological methods, symbolic dynamics,...

- **Troubles when treating complex systems**
  (1) Huge number of interacting agents (high-dimensional)
  (2) Heterogeneity in spatiotemporal scales (numerical challenge)
  (3) Hierarchical structure and great many dynamic modes
  (4) Lack of exact mathematical description
  (5) Uncertainty in data or parameters (noise or ignorance)
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Linear systems and their solution

- General form: \( \dot{x} = Ax \) with

\[
A = \begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{pmatrix}
\]
and \( x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \).

- Idea: separate solutions into independent modes by assuming \( x(t) = e^{\lambda t}v \).

- We then obtain an eigenvalue equation

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Av = \lambda v.
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Linearization of nonlinear systems

- Hartman’s theorem and Poincaré-Siegel theorem.
- Global linearization: weak nonlinearity or symmetry by lie group theory.
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Extension of Hartman’s theorem for differential flows.

- The theorem can be extended to diffeomorphic mappings or flows with periodic driving.
- Linearization around an attractive or repulsive periodic orbit.
- How to treat saddles?
  The above theorems are applicable to flows on stable or unstable manifolds.
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5 Summary
Two examples

- Consider the 1-d equation $\dot{x} = x - x^3$. The transformation
  
  $$ x = b(y) = \frac{y}{\sqrt{1 + y^2}} $$

  results in $\dot{y} = y$, valid for $x \in [-1, 1]$.

- Consider the 2-d system $\dot{z}_1 = 2z_1, \dot{z}_2 = 4z_2 + z_1^2$. The transformation
  
  $$ z_1 = y_1, z_2 = y_2 + t(y_1, y_2)y_1^2 $$

  where $t(y_1, y_2) = \frac{1}{4} \ln y_1^2$ results in

  $$ \dot{y}_1 = 2y_1, \dot{y}_2 = 4y_2. $$

[Y. Lan and I. Mezic, Physica D. 242, 42(2013)]
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Remarks on the linearization theorem

- According to Morse theory, the whole phase space of a hyperbolic system can be viewed as a gradient(-like) system modulo the minimal invariant sets. The phase space is a juxtaposition of linearizable patches.

- **Problems:**
  1. Hard to identify the linearization transformation.
  2. Works only for equilibria and periodic orbits.
  3. Hard to treat systems with conservation laws, e.g. Hamiltonian systems.
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Statistical treatment of dynamical systems:
Evolution of densities: the Perron-Frobenius Operator in analogy with the Schrödinger picture;
Evolution of observables: the Koopman operator in analogy with the Heisenberg picture.

- For a map $x_{n+1} = f(x_n)$ and a function $g(x)$, the Koopman operator $U \circ g(x) = g(f(x))$.
- For a flow $\phi(x, t)$ and a function $g(x)$, a semigroup of Koopman operators could be defined as $U_t \circ g(x) = g(\phi(x, t))$. 
Koopman operator, a way out?

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It is linear so its eigenvalues and eigenmodes are interesting objects.

Examples:
(1) For a 1-d linear map \( x_{n+1} = \lambda x_n \) and the observable \( g(x) = x^m \),

\[
U \circ g(x) = (\lambda x)^m = \lambda^m x^m = \lambda^m g(x).
\]

In fact, for \( C^1 \) observables, the most general form of the eigenfunction with eigenvalue \( \lambda^a \) is

\[
g(x) = x^a \hat{g}(\ln x),
\]

where \( \hat{g}(\cdot) \) is periodic with period \( \ln \lambda \).
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Examples:

(2) For a 1-d equation $\dot{x} = \lambda x$ and observable $g(x) = x^n$

$$U_t \circ g(x) = (xe^{\lambda t})^n = e^{n\lambda t} x^n = e^{n\lambda t} g(x).$$

In fact, for $C^1$ observables, it can be proven that the general form of the eigenfunction is just as above.

Extension to multi-dimensional linear systems and time periodic linear systems with a hyperbolic fixed point.

[Y. Lan and I. Mezić, Spectrum of the Koopman operator based on linearization, in preparation]
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Nontrivial examples

- Note that \( a(x) \) in the Hartman-Grobman’s theorem satisfies
  \[
  U_t a(x) = a \circ \phi(x, t) = e^{At} a(x)
  \]

- Suppose
  \[
  V^{-1} AV = \Lambda,
  \]
  then we get
  \[
  V^{-1} a \circ \phi(x, t) = V^{-1} e^{At} a(x),
  \]
  and so \( k = V^{-1} a \) satisfies
  \[
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  \]
  i.e. each component function of \( k \) is an eigenfunction of \( U_t \).
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Construction of eigenmodes along trajectories

- For the map $x_{n+1} = T(x_n)$ and function $g(x)$, consider

$$g^{*}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} g(T^j x),$$

which is an eigenfunction of the Koopman operator with eigenvalue 1.

- Furthermore, the construction

$$g^{\omega}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{i2\pi j \omega} g(T^j x)$$

defines an eigenfunction of the Koopman operator with eigenvalue $e^{-i2\pi \omega}$.
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Connection with Liouville operator

- In a Hamiltonian system with $H(p,q)$, the Liouville operator $L$ may be defined as

$$L \circ f(x) = -i \sum_j \left[ \frac{\partial H}{\partial p_j} \frac{\partial f}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial f}{\partial p_j} \right],$$

which could be written with the Poisson bracket $L \circ f(x) = -i[f,H]$.

- It is related to Koopman operator by the exponentiation $U_t = \exp(itL)$.

- On a transitive invariant set, Koopman operator is unitary, i.e.

$$U_t g(x) = e^{it\alpha} g(x) = e^{i(t\alpha + \arg(g))} |g(x)|.$$

$|g(x)|$ is invariant and the phase increases linearly. Hence, they constitute the action-angle variable.
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Spectral decomposition of evolution equations

For an evolution equation in an infinite-dimensional Hilbert space $v(x)^{n+1} = N(v(x)^n, p)$ and if the attractor $M$ is of finite dimension with the evolution $m^{n+1} = T(m^n)$.

For an observable $g(x, m)$, we have

$$Ug(x, m) = U_s g(x, m) + U_r g(x, m)$$

$$= g^*(x) + \sum_{j=1}^k \lambda_j f_j(m) g_j(x) + \int_0^1 e^{i2\pi\alpha} dE(\alpha) g(x, m).$$

- $U_s$: the singular part of the operator corresponding to the discrete part of the spectrum, viewed as a deterministic part.
- $U_r$: the regular part of the operator corresponding to the continuous part of the spectrum, modeled as a stochastic process.

[I. Mezic, Nonlinear Dynamics 41, 309(2005)]
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The Chirikov standard map is

\[ x_1^* = x_1 + 2\pi \epsilon \sin(x_2) \mod 2\pi \]

\[ x_2^* = x_1^* + x_2 \mod 2\pi \]

The embedding of dynamics into space of three observables.

[M. Budisic and I. Mezic, 48th IEEE Conference on Decision and Control]
Organizing invariant set by diffusion map

Eigenvalues for $\lambda_1, \lambda_2, \lambda_7, \lambda_{17}$ at $\epsilon = 0.133$. 
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Fluid dynamics

- The Navier-Stokes equation

\[ \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v} \]

with \( \nabla \cdot \mathbf{v} = 0 \) describes incompressible Newtonian fluids.

- With different Reynold’s number \( Re = Lu/\nu \), the system experience a series of bifurcation: laminar \( \rightarrow \) periodic \( \rightarrow \) turbulent

- Turbulence is a spatiotemporal chaos with enormous space-time structures and scales.

- Jet in cross flow: turbulent but with large eddies. Could we describe it with the Koopman operator approach?
Fluid dynamics

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Jet in cross flow

The Arnoldi algorithm

- Consider a linear dynamical system $x_{k+1} = Ax_k$ and construct the matrix

$$K = [x_0, x_1, \ldots, x_{m-1}] = [x_0, Ax_0, \ldots, A^{m-1}x_0].$$

If the $m$th iterate $x_m = Ax_{m-1} = \sum_{k=0}^{m-1} c_k x_k + r$, the we can write $AK \approx KC$, where

$$C = \begin{pmatrix}
0 & 0 & 0 & \cdots & c_0 \\
1 & 0 & 0 & \cdots & c_1 \\
0 & 1 & 0 & \cdots & c_2 \\
& \cdots & \cdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & c_0
\end{pmatrix}.$$  

- If $Ca = \lambda a$, then the value $\lambda$ and the vector $v = Ka$ are approximate eigenvalue and eigenvector of the original matrix $A$. 

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The Arnoldi algorithm

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\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & c_0
\end{pmatrix}.$$

- If $Ca = \lambda a$, then the value $\lambda$ and the vector $v = Ka$ are approximate eigenvalue and eigenvector of the original matrix $A$. 
Two structure functions

The two eigenmodes
Summary

- Linearization is possible in the basin of attraction of a hyperbolic set.
- Koopman operator provides a way to identify the linearization transformation.
- On the minimal invariant set, the spectrum of the Koopman operator is on the unit circle.
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