Renormalization group: φ^4-theory and ε-expansion

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Outline

- Ising model: from lattice to continuous field theory
- Dimensional analysis, power counting, UV cut-off
- Critical dimension, relevant-irrelevant operators
- Renormalization group: the Gaussian model
- Renormalization group: cumulant expansion
- The fixed points: Gaussian and non-trivial
- $\mathcal{E}$-expansion, order one and higher orders.
- Relation to Renormalization theory in QFT

Ising model: From lattice model to continuous Field Theory

- Ising model: magnetic spins $S_i = \pm 1$ along the *z*-direction, in a magnetic field

- Short range interactions $J_{ij} = J(r_i - r_j)$ between any pair of spins *i* and *j* . We study a ferromagnetic system $J_{ij} < 0$ (e.g. nearest neighbor interactions)

- Partition function $Z = \sum_{S_i = \pm 1} e^{\frac{\beta}{2} \sum_{i,j} K_{ij} S_i S_j + \beta H \sum_i S_i}$

$$K_{ij} = -J_{ij} > 0$$

- Gaussian integral

$$\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i A_{ij} x_j + u_i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} e^{\frac{1}{2}u_i A^{-1}_{ij} u_j}$$

- so

$$e^{\frac{\beta}{2}K_{ij}S_i S_j} \sim \int \prod_{i=1}^{N} d\phi_i e^{-\frac{\beta}{2}\phi_i K^{-1}_{ij}\phi_j + \beta\phi_i S_i}$$

- Partition function

$$Z = \sum_{S_i=\pm 1} \int \prod_{i=1}^{N} d\phi_i e^{-\frac{\beta}{2}\phi_i K^{-1}_{ij}\phi_j} e^{\beta \sum_i (H+\phi_i) S_i}$$

$$= \int \prod_{i=1}^{N} d\phi_i e^{-\frac{\beta}{2}\phi_i K^{-1}_{ij}\phi_j + \sum_i \log 2\cosh \beta(H+\phi_i)}$$

$$K(k) = \int d^d r e^{ikr} K(r)$$

microscopic scale

$$K^{-1}(k) = \frac{1}{K(k)}$$

macroscopic distance

Large distance behavior $k \to 0$ $\qquad r \gg a$

$$K(k) \sim_{k\to 0} K_0(1 - ck^2) \qquad c > 0$$

$$K(k)^{-1} \sim_{k\to 0} K_0^{-1}(1 + ck^2)$$

$$\sum_{i,j} \phi_i K_{ij}^{-1} \phi_j \sim K_0^{-1} \int \frac{d^d k}{(2\pi)^d} \tilde{\phi}(k)(1 + ck^2 + \ldots)\tilde{\phi}(-k)$$

The continuous limit

- If we are interested in correlation functions at distances large compared to lattice spacing (near a critical point), we can take a continuous limit.

- Exercise: shift field by magnetic field, redefine coefficients and rescale fields, expand *log cosh,* and show that the partition function can be written as

$$Z = \int \mathcal{D}\phi(r) e^{-\int d^d r \left(\frac{1}{2}\phi(r)(-\nabla^2 + r_0)\phi(r) + h\phi(r) + \sum_{m=2}^{\infty} u_{2m}\phi^{2m}(r) \right)}$$

proportional to $T - T_c$ in mean-field

Dimensional analysis and power counting: canonical dimensions

- All the analysis and integrals will be done in momentum space (*k*-space) so do dimensional analysis in momentum space $\Lambda = \frac{2\pi}{a}$ UV cut-off

$$\int d^d r (\nabla \phi)^2 = 1 = \Lambda^{-d+2} \Lambda^{2[\phi]} \Rightarrow [\phi] = \frac{d-2}{2}$$

- It follows that

$$[h] = \frac{d+2}{2} > 0$$

$$[r_0] = 2$$

$$[u_0] = [u_4] = 4 - d$$

$$[u_{2m}] = d - (d-2)m$$

- As we will see, an operator is relevant, marginal, irrelevant if its dimension δ is $\delta > 0,\ \delta = 0,\ \delta < 0$

- The magnetic field is relevant in all dimensions.

- The “mass” r_0 is always relevant

- In *d=2,* all operators are relevant

- In *d=3,* only ϕ^4 relevant, ϕ^6 marginal

- In *d>3,* only ϕ^4 relevant

The ϕ^4 field-theory for critical phenomena

- As we will discuss a specific scheme of calculation (epsilon expansion), we will work around dimension 4. Then only one relevant operator. Near criticality, near dimension 4, the system is thus well described by

$$Z = \int \mathcal{D}\phi(r) e^{-\int d^d r \left(\frac{1}{2}\phi(r)(-\nabla^2 + r_0)\phi(r) + \frac{u_0}{4!}\phi^4(r) + h\phi(r) \right)}$$

ϕ^4 field-theory

- Note that r_0 is the difference between terms from

$K(k)$ and from the *log cosh,* so it can change sign

Renormalization group: the Gaussian model

- Near criticality, correlation length $\xi \gg a$
- System is (nearly) scale invariant
- Equivalent of block spins: momentum shell integration
- Originally, UV cut-off $\Lambda = \dfrac{2\pi}{a}$
- Define $\Lambda' = \Lambda - d\Lambda$
- (Try to) integrate over the fields $\Lambda' < k < \Lambda$
- Rescale momenta from Λ' to Λ

The Gaussian Model

Assume $u_0 = 0$

$$Z = \int \mathcal{D}\phi(r) e^{-\int d^d r \left(\frac{1}{2}\phi(r)(-\nabla^2 + r_0)\phi(r) + h\phi(r)\right)}$$

$$Z = \int \mathcal{D}\phi(k) \exp\left(-\int_0^\Lambda \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2}\phi(k)(k^2 + r_0)\phi(-k)\right) + h\phi(k=0)\right)$$

$$\phi_<(k) = \phi(k) \;\; \text{iff} \;\; 0 < k < \Lambda'$$, 0 otherwise

$$\phi_>(k) = \phi(k) \;\; \text{iff} \;\; \Lambda' < k < \Lambda$$, 0 otherwise

$$\phi(k) = \phi_<(k) + \phi_>(k)$$

$$\phi_<(k)\phi_>(k) = 0$$

Note: the model does not have a low T phase

$$Z = Z_1 \int \mathcal{D}\phi_<(k) \exp\left(- \int_0^{\Lambda'} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2}\phi_<(k)(k^2 + r_0)\phi_<(-k) \right) + h\phi_<(k=0) \right)$$

$$Z_1 = \int \mathcal{D}\phi_>(k) \exp\left(- \int_{\Lambda'}^{\Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2}\phi_>(k)(k^2 + r_0)\phi_>(-k) \right) \right)$$

constant

Rescaling of *k* $\longrightarrow q = \dfrac{\Lambda}{\Lambda'} k$

$$\int_0^{\Lambda'} d^d k k^2 \phi_<(k)\phi_<(-k) = \left(\frac{\Lambda'}{\Lambda} \right)^{d+2} \int_0^{\Lambda} d^d q \; q^2 \phi_<(q)\phi_<(-q)$$

Rescaling of $\phi_<$ so that coefficient is 1/2: $\phi = \left(\dfrac{\Lambda'}{\Lambda} \right)^{\frac{d+2}{2}} \phi_<$

$$\frac{\Lambda'}{\Lambda} = 1 - \frac{d\Lambda}{\Lambda}$$
$$= 1 - dl$$

$$Z = Z_1 \int \mathcal{D}\phi(k) \exp\left(- \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2}\phi(k)\left(k^2 + \left(\frac{\Lambda'}{\Lambda}\right)^{-2} r_0\right)\phi(-k)\right) + \left(\frac{\Lambda'}{\Lambda}\right)^{-\frac{d+2}{2}} h\phi(k=0)\right)$$

System equivalent to original one, for large distances

$$r_0' = (1 - dl)^{-2} r_0$$
$$h' = (1 - dl)^{-\frac{d+2}{2}} h$$

Differential form = Flow equations

$$\frac{dr}{dl} = 2r$$
$$\frac{dh}{dl} = \frac{d+2}{2} h$$

Canonical dimensions of operators

Relevant operators. Fixed points

$$r^* = 0$$
$$h^* = 0$$

Compute exponents!

Cumulant expansion

Include the interaction term

$$Z = \int \mathcal{D}\phi(r) e^{-\int d^d r \left(\frac{1}{2}\phi(r)(-\nabla^2 + r_0)\phi(r) + \frac{u_0}{4!}\phi^4(r) + h\phi(r) \right)}$$

Take *h=0*

In Fourier space

$$Z = \int \mathcal{D}\phi(k) \exp\left(-\int_0^\Lambda \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2}\phi(k)(k^2 + r_0)\phi(-k) \right) \right.$$

$$\left. + \frac{u_0}{4!}(2\pi)^d \int_0^\Lambda \prod_{i=1}^4 \frac{d^d k_i}{(2\pi)^d} \delta^{(d)}(k_1 + k_2 + k_3 + k_4)\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4) \right)$$

Make separation

$$\phi = \phi_< + \phi_>$$

$$Z = \int \mathcal{D}\phi_<(r) \exp\left(-\int d^d r \Big(\frac{1}{2}\phi_<(r)(-\nabla^2 + r_0)\phi_<(r) + \frac{u_0}{4!}\phi_<^4(r)\Big)\right)$$
$$\times \int \mathcal{D}\phi_>(r) \exp\left(-\int d^d r \Big(\frac{1}{2}\phi_>(r)(-\nabla^2 + r_0)\phi_>(r) + \frac{u_0}{4!}(\underbrace{4\phi_<^3\phi_>}_{1} + \underbrace{6\phi_<^2\phi_>^2}_{2} + \underbrace{4\phi_<\phi_>^3}_{3} + \underbrace{\phi_>^4}_{4})\Big)\right)$$

In the high temperature phase (no symmetry breaking), do a cumulant expansion (equivalent to a loop expansion) to integrate over $\phi_>$

$$Z(\phi_<) = \int \mathcal{D}\phi_> \exp\left(-\int d^d r (H_0(\phi_>) + V(\phi_<, \phi_>))\right)$$
$$= Z_0 \langle \exp\left(-\int d^d r V(\phi_<, \phi_>)\right)\rangle_0$$

$$Z_0 = \int \mathcal{D}\phi_> \exp\left(-\int d^d r H_0(\phi_>)\right)$$

$$\langle W(\phi_>)\rangle_0 = \frac{1}{Z_0}\int \mathcal{D}\phi_> \exp\left(-\int d^d r H_0(\phi_>)\right) W(\phi_>)$$

Expanding to the second cumulant

$$Z = Z_0 \exp\left(- \int d^d r \langle V(\phi_>(r)) \rangle_0 + \frac{1}{2} \int d^d r d^d r' \Big(\langle V(\phi_>(r)) V(\phi_>(r')) \rangle_0 - \langle V(\phi_>(r)) \rangle_0 \langle V_>(\phi(r')) \rangle_0 \Big) \right)$$

$$\int d^d r H_0(\phi_>) = \frac{1}{2} \int d^d r \ \phi_>(r)(-\nabla^2 + r_0)\phi_>(r)$$
$$= \frac{1}{2} \int_{\Lambda'}^{\Lambda} \frac{d^d k}{(2\pi)^d} \ \phi_>(k)(k^2 + r)\phi_>(-k)$$

$$\int d^d r V(\phi_>) = \frac{u_0}{4!} \int d^d r \left(\underbrace{4\phi_<^3 \phi_>}_{1} + \underbrace{6\phi_<^2 \phi_>^2}_{2} + \underbrace{4\phi_< \phi_>^3}_{3} + \underbrace{\phi_>^4}_{4} \right)$$
$$= \frac{u_0}{4!} \int_{\Lambda'}^{\Lambda} \prod_{i=1}^{4} \frac{d_i^k}{(2\pi)^d} (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4) c_\alpha \phi_\alpha(k_1)\phi_\alpha(k_2)\phi_\alpha(k_3)\phi_\alpha(k_4)$$

coeff. 1,4,6

Wick Theorem

$$\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i A_{ij} x_j + u_i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} e^{\frac{1}{2}u_i A^{-1}_{ij} u_j}$$

By taking derivatives w.r.t u_i and then setting $u_i = 0$

$$\langle x_i x_j \rangle = A^{-1}_{ij}$$

$\langle x_{i_1} x_{i_2} \ldots x_{i_m} \rangle = 0$ if *m* odd

$$\langle x_{i_1} x_{i_2} \ldots x_{i_m} \rangle = \sum_{\text{all complete sets of pairings}} A^{-1}_{i_1 i_{P1}} A^{-1}_{i_2 i_{P2}} \ldots A^{-1}_{i_n i_{Pn}}$$

if *m* even

Example: $\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$\langle x^2 \rangle = 1$ Wick $\langle x^{2n} \rangle = (2n-1)!! = (2n-1).(2n-3)\ldots 3.1$

Cumulant expansion

$$\langle \phi_>(k)\phi_>(k')\rangle = \frac{\delta^{(d)}(k-k')}{k^2+r_0}$$

Use Wick theorem

Order 1 in *V:* only term 2 contributes to r_0

$$\langle \phi_>^2(r)\rangle = \int_{\Lambda'}^{\Lambda} \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r_0}$$
$$= S_d \frac{\Lambda^{d-1}}{(2\pi)^d}\frac{d\Lambda}{\Lambda^2+r_0}$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$

Order 2 in *V:* only term 2-2 contributes to u_0

$$\frac{1}{2}\left(\frac{u_0}{4!}\right)^2 \times\ 36\int d^drd^dr'\phi_<^2(r)\phi_<^2(r')\left(\langle\phi_>^2(r)\phi_>^2(r')\rangle - \langle\phi_>^2(r)\rangle\langle\phi_>^2(r')\rangle\right)$$

$$\frac{1}{2}\left(\frac{u_0}{4!}\right)^2 \times 2 \times 36 \int d^d r d^d r' \phi_<^2(r)\phi_<^2(r')\langle\phi_>(r)\phi_>(r')\rangle^2$$

$$g(r-r') = \langle\phi_>(r)\phi_>(r')\rangle = \int_{\Lambda'}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{e^{ik(r-r')}}{k^2+r_0}$$

Propagator *g* is short ranged (because large *k*): expansion in gradients

$$\begin{aligned}
g^2(r) &= \gamma_0 \delta^{(d)}(r) + \ldots \\
\gamma_0 &= \int d^d r g^2(r) \\
&= \int_{\Lambda'}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+r_0)^2} \\
&= S_d \frac{\Lambda^{d-1} d\Lambda}{(2\pi)^d} \frac{1}{(\Lambda^2+r_0)^2}
\end{aligned}$$

$$r_0' = r_0 + \frac{u_0}{2} S_d \frac{\Lambda^{d-1} d\Lambda}{(2\pi)^d (\Lambda^2 + r_0)}$$

$$u_0' = u_0 - \frac{3}{2} u_0^2 S_d \frac{\Lambda^{d-1} d\Lambda}{(2\pi)^d (\Lambda^2 + r_0)^2}$$

$$Z = C \int \mathcal{D}\phi_<(r) \exp\left(- \int_{\Lambda'} d^d r \Big(\frac{1}{2} \phi_<(r)(-\nabla^2 + r_0')\phi_<(r) + \frac{u}{4!} \phi_<^4(r) \Big) \right)$$

Remains to do: 1) rescale momenta from Λ' to Λ

2) rescale $\phi_<$ so that coefficient equal 1/2

Renormalization Group Equations for running coupling constants $(r(l), u(l))$

$$\frac{dr}{dl} = 2r + \frac{u}{2} \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{(\Lambda^2 + r)}$$

$$\frac{du}{dl} = \varepsilon u - \frac{3}{2} u^2 \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{(\Lambda^2 + r)^2}$$

$$\varepsilon = 4 - d$$

Note that $\quad \frac{dh}{dl} = \frac{2+d}{2} h$

Fixed points:
The epsilon expansion

Gaussian: $r^* = u^* = h^* = 0$

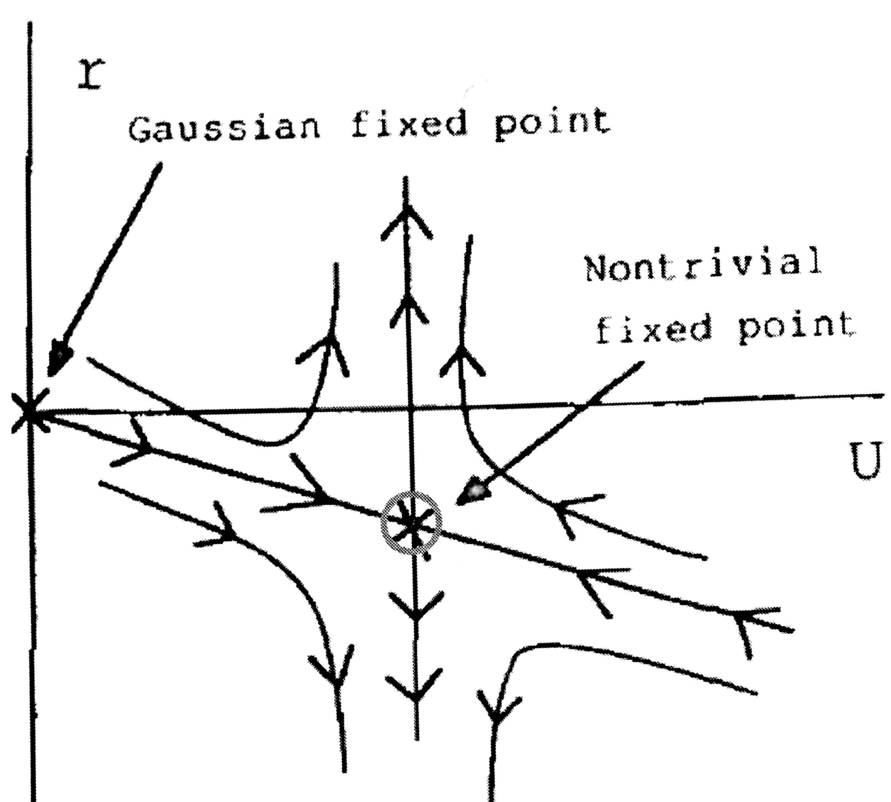
Non trivial: $h^* = 0, \quad r^* \text{ and } u^* \text{ of order } \varepsilon$

To order 1 in ε

$$r^* = -\frac{\Lambda^2}{6}\varepsilon$$

$$u^* = \frac{16\pi^2}{3}\varepsilon$$

Linearize equations around non trivial fixed point:



relevant

$$y_1 = 2 - \frac{\varepsilon}{3} > 0$$

$$y_2 = -\varepsilon < 0$$

irrelevant

Critical exponents to order one in epsilon

$$\alpha = \frac{\varepsilon}{6}$$

$$\beta = \frac{1}{2} - \frac{\varepsilon}{6}$$

$$\gamma = 1 + \frac{\varepsilon}{6}$$

$$\delta = 3 + \varepsilon$$

$$\eta = 0$$

$$\varepsilon = 4 - d$$

- Relation to Renormalization theory in QFT