Homogeneous relaxation models & methods
for
compressible two-phase flow
II: Relaxation scheme & stiff solver

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Outline

Cavitating Richtmyer-Meshkov problem

Gas volume fraction
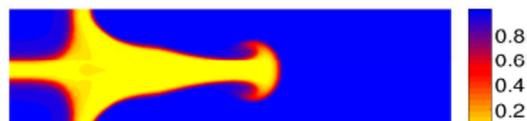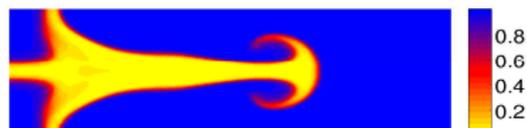
Mixture pressure
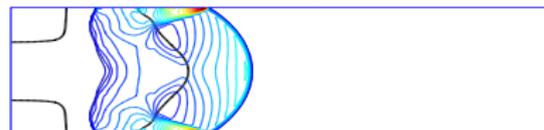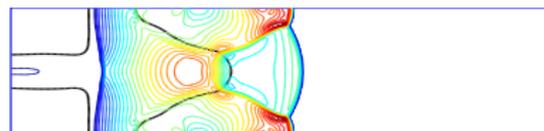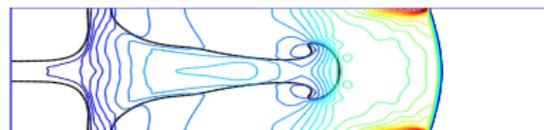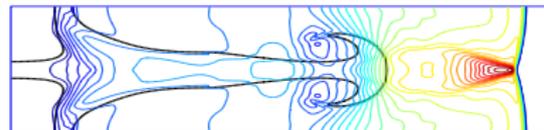
High-speed underwater projectile

With thermo-chemical relaxation

No thermo-chemical relaxation

Vapor volume fraction

Vapor mass fraction

Mixture density

Mixture pressure
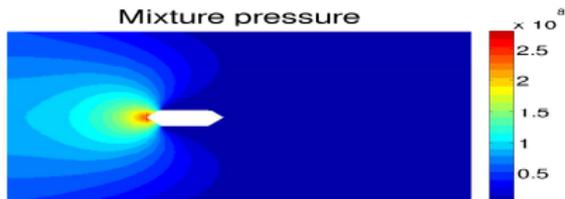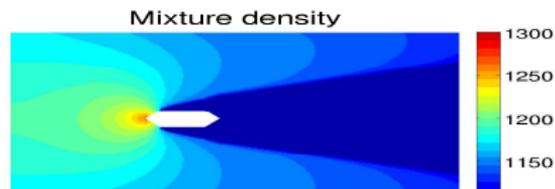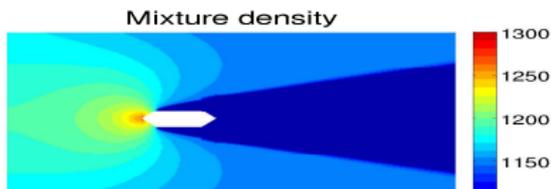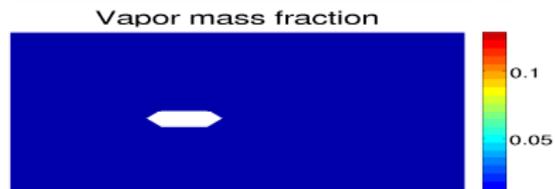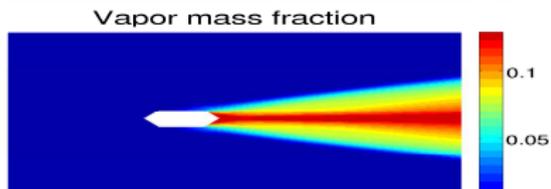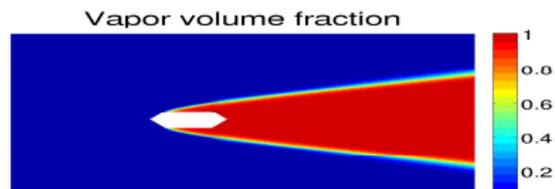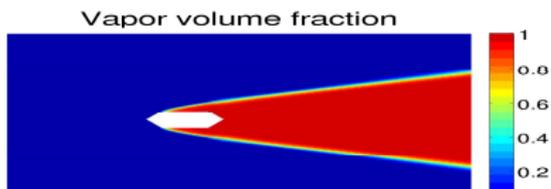
High-pressure fuel injector



With thermo-chemical relaxation

No thermo-chemical relaxation

Vapor volume fraction

Vapor volume fraction

Vapor mass fraction

Vapor mass fraction

Mixture density

Mixture density

Mixture pressure

Mixture pressure

Vapor temperature

Vapor temperature
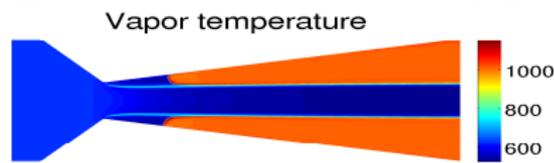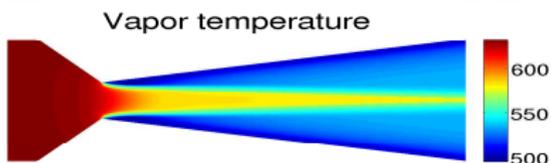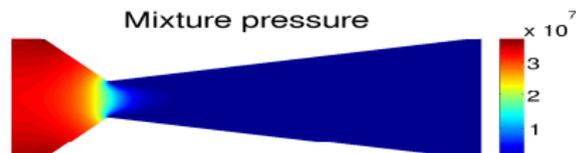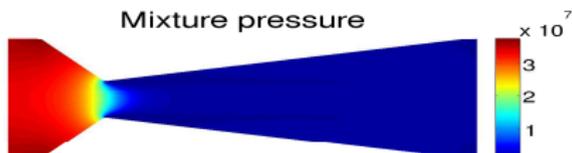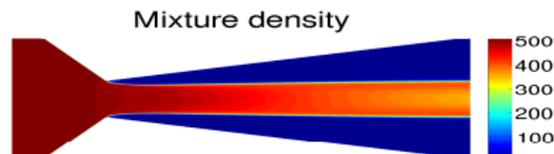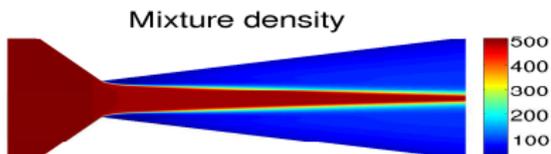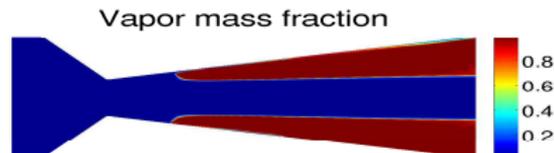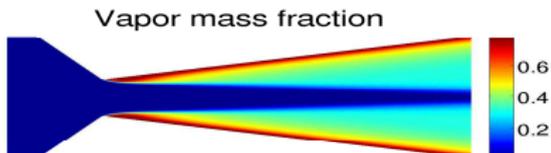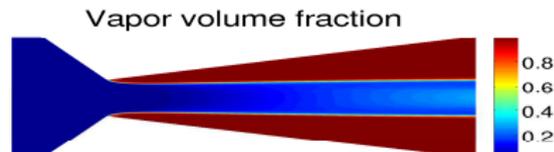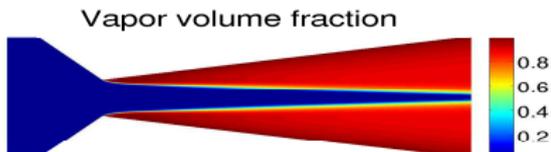
Phasic-total-energy-based HRM

Consider 1-velocity phasic-total-energy based homogeneous relaxation model

$$\partial_t (\alpha_1 \rho_1) + \nabla \cdot (\alpha_1 \rho_1 \vec{u}) = \nu (g_2 - g_1)$$

$$\partial_t (\alpha_2 \rho_2) + \nabla \cdot (\alpha_2 \rho_2 \vec{u}) = \nu (g_1 - g_2)$$

$$\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla (\alpha_1 p_1 + \alpha_2 p_2) = 0$$

$$\partial_t (\alpha_1 \rho_1 E_1) + \nabla \cdot (\alpha_1 \rho_1 E_1 \vec{u} + \alpha_1 p_1 \vec{u}) + \Sigma (w, \nabla w) =$$
$$\mu p_I (p_2 - p_1) + \theta T_I (T_2 - T_1) + \nu g_I (g_2 - g_1)$$

$$\partial_t (\alpha_2 \rho_2 E_2) + \nabla \cdot (\alpha_2 \rho_2 E_2 \vec{u} + \alpha_2 p_2 \vec{u}) - \Sigma (w, \nabla w) =$$
$$\mu p_I (p_1 - p_2) + \theta T_I (T_1 - T_2) + \nu g_I (g_1 - g_2)$$

$$\partial_t \alpha_1 + \vec{u} \cdot \nabla \alpha_1 = \mu (p_1 - p_2) + \nu v_I (g_1 - g_2)$$

$$\Sigma = \vec{u} \cdot [Y_1 \nabla (\alpha_2 p_2) - Y_2 \nabla (\alpha_1 p_1)], \quad Y_k = \frac{\alpha_k \rho_k}{\rho}$$

Closure model: Stiffened gas EOS (linear EOS)

HRM model in compact form

$$\partial_t w + \nabla \cdot f(w) + \mathcal{B}\left(w, \nabla w\right) = \psi_\mu(w) + \psi_\theta(w) + \psi_\nu(w)$$

where

$$w = \left[\alpha_1, \ \alpha_1\rho_1, \ \alpha_2\rho_2, \ \rho\vec{u}, \ \alpha_1\rho_1 E_1, \ \alpha_2\rho_1 E_2, \ \alpha_1\right]^T$$

$$f = \left[\alpha_1\rho_1\vec{u}, \ \alpha_2\rho_2\vec{u}, \ \rho\vec{u} \otimes \vec{u} + (\alpha_1 p_1 + \alpha_2 p_2)\overline{\overline{I}}, \right.$$
$$\left. \alpha_1\left(\rho_1 E_1 + p_1\right)\vec{u}, \ \alpha_2\left(\rho_2 E_2 + p_2\right)\vec{u}, \ 0\right]^T$$

$$\mathcal{B} = \left[0, \ 0, \ 0, \ \Sigma\left(w, \nabla w\right), \ -\Sigma\left(w, \nabla w\right), \ \vec{u} \cdot \nabla\alpha_1\right]^T$$

$$\psi_\mu = \left[0, \ 0, \ 0, \ \mu p_I\left(p_2 - p_1\right), \ \mu p_I\left(p_1 - p_2\right), \ \mu\left(p_1 - p_2\right)\right]^T$$

$$\psi_\theta = \left[0, \ 0, \ 0, \ \theta T_I\left(T_2 - T_1\right), \ \theta T_I\left(T_1 - T_2\right), \ 0\right]^T$$

$$\psi_\nu = \left[\nu\left(g_2 - g_1\right), \ \nu\left(g_1 - g_2\right), \ 0, \ \nu g_I\left(g_2 - g_1\right), \right.$$
$$\left. \nu g_I\left(g_1 - g_2\right), \ \nu v_I\left(g_1 - g_2\right)\right]^T$$

Relaxation scheme

To find approximate solution of HRM, in each time step, fractional-step method is employed:

1. Non-stiff hyperbolic step
   Solve hyperbolic system without relaxation sources

   $$\partial_t w + \nabla \cdot f(w) + w\,(w, \nabla w) = 0$$

   using state-of-the-art solver over time interval Δt

2. Stiff relaxation step
   Solve system of ordinary differential equations

   $$\partial_t w = \psi_\mu(w) + \psi_\theta(w) + \psi_\nu(w)$$

   in various flow regimes under relaxation limits

Definition (mixture-energy consistent)

(i) Mixture total energy conservation consistency

$$(\rho E)^0 = (\rho E)^{0,\mathrm{C}}$$

where $(\rho E)^0 = (\alpha_1 \rho_1 E_1)^0 + (\alpha_2 \rho_2 E_2)^0$

(ii) Relaxed pressure consistency

$$(\rho e)^{0,\mathrm{C}} = \alpha_1^* \rho_1 e_1 \left(p^*, \frac{(\alpha_1 \rho_1)^0}{\alpha_1^*} \right) + \alpha_2^* \rho_2 e_2 \left(p^*, \frac{(\alpha_2 \rho_2)^0}{\alpha_2^*} \right),$$

where $(\rho e)^{0,\mathrm{C}} = (\rho E)^{0,\mathrm{C}} - \frac{(\rho \vec{u})^0 \cdot (\rho \vec{u})^0}{2 \rho^0}$

Method proposed here with phasic-total-energy formulation is mixture-energy consistent

Relaxation scheme: Stiff solvers

1. **Algebraic-based** approach
   - Saurel *et al.* (JFM 2008), Zein *et al.* (JCP 2010), LeMartelot *et al.* (JFM 2013), Pelanti-Shyue (JCP 2014)
   - Impose equilibrium conditions directly, without making explicit of interface states p_I, g_I,...

2. **Differential-based** approach
   - Saurel *et al.* (JFM 2008), Zein *et al.* (JCP 2010)
   - Impose differential of equilibrium conditions, require explicit of interface states p_I, g_I,...

3. **Optimization-based** approach (for mass transfer only)
   - Helluy & Seguin (ESAIM: M2AN 2006), Faccanoni *et al.* (ESAIM: M2AN 2012)

p relaxation

Assume frozen thermal & thermo-chemical relaxation, *i.e.*, $\theta = 0$ & $\nu = 0$, look for solution of ODEs in limit $\mu \to \infty$

$$\partial_t \left(\alpha_1 \rho_1 \right) = 0$$
$$\partial_t \left(\alpha_2 \rho_2 \right) = 0$$
$$\partial_t \left(\rho \vec{u} \right) = 0$$
$$\partial_t \left(\alpha_1 \rho_1 E_1 \right) = \mu p_I \left(p_2 - p_1 \right)$$
$$\partial_t \left(\alpha_2 \rho_2 E_2 \right) = \mu p_I \left(p_1 - p_2 \right)$$
$$\partial_t \alpha_1 = \mu \left(p_1 - p_2 \right)$$

Under mechanical equilibrium with equal pressure

$$p_1 = p_2 = p$$

We find easily

$$\alpha_k \rho_k = \alpha_{k0}\rho_{k0}, \quad \rho = \rho_0, \quad \vec{u} = \vec{u}_0, \quad \rho E = (\rho E)_0, \quad e = e_0$$

$$\partial_t (\alpha \rho E)_k = \partial_t (\alpha \rho e)_k = -p_I \partial_t \alpha_k, \qquad k = 1, 2$$

We find easily

$$\alpha_k \rho_k = \alpha_{k0} \rho_{k0}, \quad \rho = \rho_0, \quad \vec{u} = \vec{u}_0, \quad \rho E = (\rho E)_0, \quad e = e_0$$
$$\partial_t \left(\alpha \rho E \right)_k = \partial_t \left(\alpha \rho e \right)_k = -p_I \partial_t \alpha_k, \qquad k = 1, 2$$

Integrating latter equation & using $\alpha_k \rho_k = \alpha_{k0} \rho_{k0}$ leads to

$$e_k \left(p_k, \rho_k \right) - e_{k0} + \bar{p}_I \left(\frac{1}{\rho_k} - \frac{1}{\rho_{k0}} \right) = 0$$

This gives condition for ρ_k in p, $k = 1, 2$, if assume *e.g.*, $\bar{p}_I = (p_I^0 + p)/2$, & impose mechanical equilibrium in EOS

Combining that with saturation condition for volume fraction

$$\frac{\alpha_1 \rho_1}{\rho_1(p)} + \frac{\alpha_2 \rho_2}{\rho_2(p)} = 1$$

leads to algebraic equation (quadratic one with SG EOS) for relaxed pressure p

With that, ρ_k, α_k can be determined & state vector w is updated from current time to next

Combining that with saturation condition for volume fraction

$$\frac{\alpha_1 \rho_1}{\rho_1(p)} + \frac{\alpha_2 \rho_2}{\rho_2(p)} = 1$$

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With that, ρ_k, α_k can be determined & state vector w is updated from current time to next

Relaxed solution depends strongly on initial condition from non-stiff hyperbolic step

Dodecane 2-phase Riemann problem: p relaxation

Mechanical-equilibrium solution at $t = 473\mu$s

density (kg/m^3)
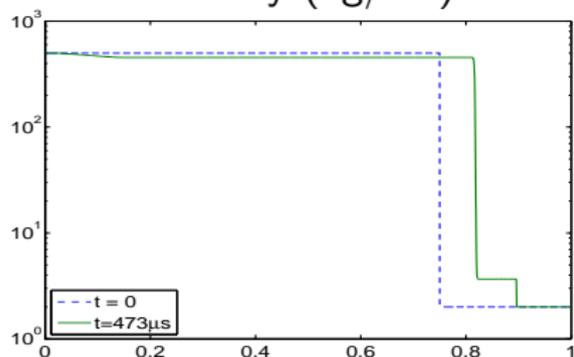
velocity (m/s)
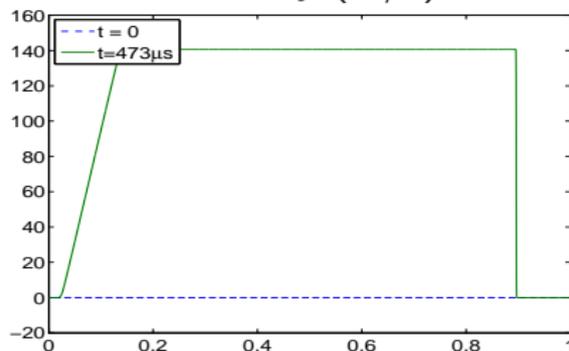
pressure(bar)
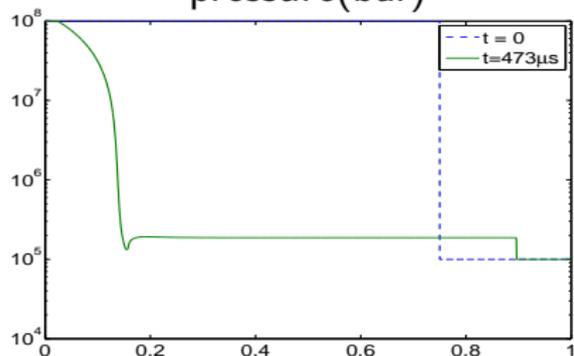
vapor volume fraction
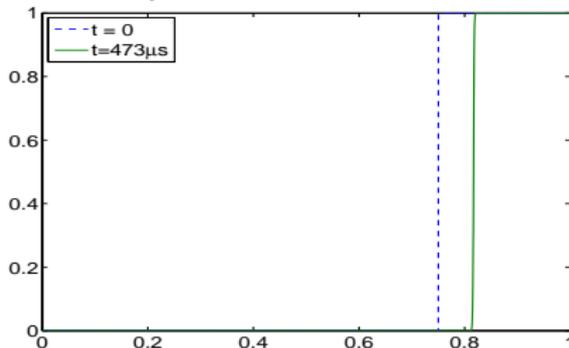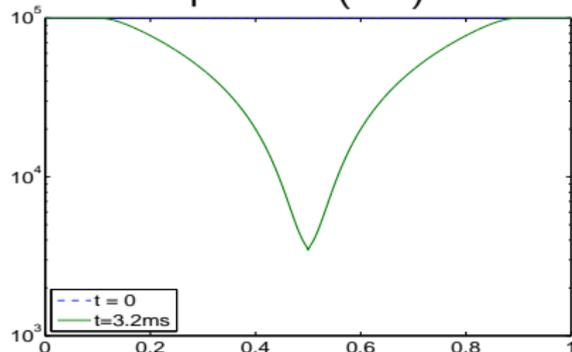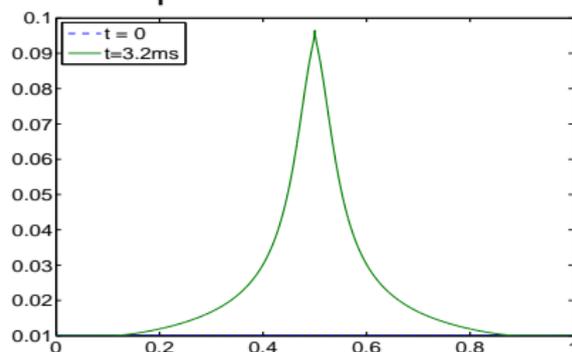
Thermodynamic path after p-relaxation in p-v phase diagram

Thermodynamic path comparison between solutions after p- & pTg-relaxation in p-v phase diagram

Mechanical-equilibrium solution at $t = 3.2$ms

Thermodynamic path comparison between solutions after p- & pTg-relaxation in p-v phase diagram
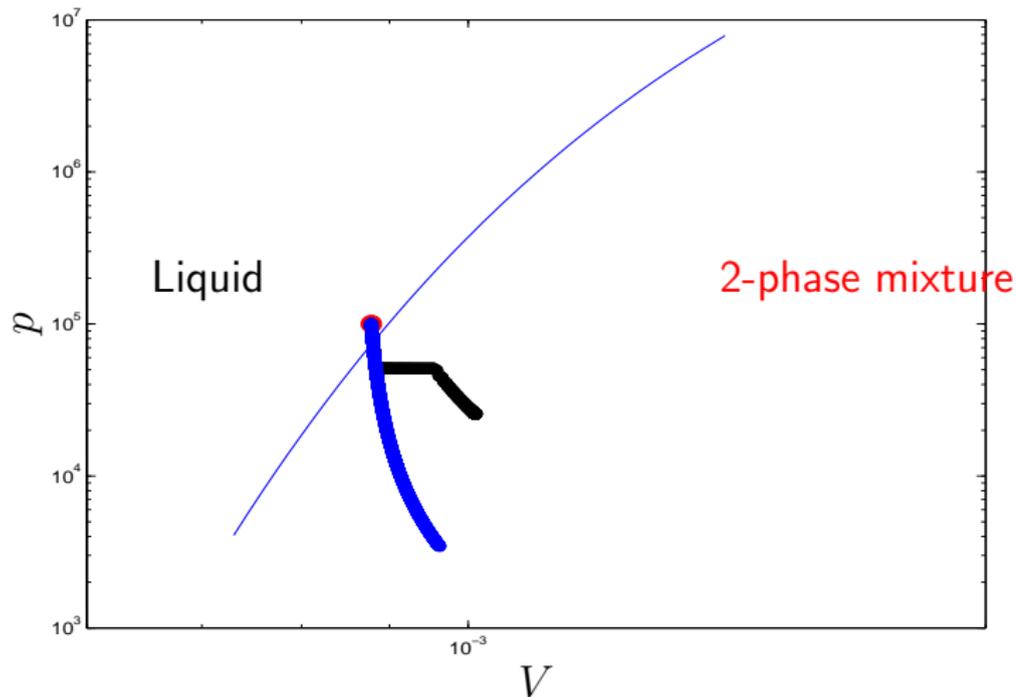
# pT relaxation

Now assume frozen thermo-chemical relaxation $\nu = 0$, look for solution of ODEs in limits $\mu \,\&\, \theta \to \infty$

$$\partial_t \left(\alpha_1 \rho_1 \right) = 0$$
$$\partial_t \left(\alpha_2 \rho_2 \right) = 0$$
$$\partial_t \left(\rho \vec{u} \right) = 0$$
$$\partial_t \left(\alpha_1 \rho_1 E_1 \right) = \mu p_I \left(p_2 - p_1 \right) + \theta T_I \left(T_2 - T_1 \right)$$
$$\partial_t \left(\alpha_2 \rho_2 E_2 \right) = \mu p_I \left(p_1 - p_2 \right) + \theta T_I \left(T_1 - T_2 \right)$$
$$\partial_t \alpha_1 = \mu \left(p_1 - p_2 \right)$$

Under mechanical-thermal equilibrium conditons

$$p_1 = p_2 = p$$
$$T_1 = T_2 = T$$

We find easily

$$\partial_t\left(\alpha_1\rho_1\right) = 0 \qquad \Longrightarrow \qquad \alpha_1\rho_1 = \alpha_1^0\rho_1^0$$

$$\partial_t\left(\alpha_2\rho_2\right) = 0 \qquad \Longrightarrow \qquad \alpha_2\rho_2 = \alpha_2^0\rho_2^0$$

$$\partial_t\left(\rho\vec{u}\right) = 0 \qquad \Longrightarrow \qquad \rho\vec{u} = \rho^0\vec{u}^0$$

$$\partial_t\left(\alpha_k\rho_k E_k\right) = \frac{\theta}{q_I}\left(T_2 - T_1\right) \quad \Longrightarrow \quad \partial_t\left(\alpha\rho e\right)_k = q_I\partial_t\alpha_k$$

Integrating latter two equations with respect to time

$$\int \partial_t\left(\alpha\rho e\right)_k \ dt = \int q_I\partial_t\alpha_k \ dt$$

$$\Longrightarrow \qquad \alpha_k\rho_k e_k - \alpha_k^0\rho_k^0 e_k^0 = -\bar{q}_I\left(\alpha_k - \alpha_k^0\right)$$

Take $\bar{q}_I = (q_I^0 + q_I)/2$ or q_I, for example, & find algebraic equation for α_1, by imposing

$$T_2\left(e_2, \alpha_2^0\rho_2^0/(1-\alpha_1)\right) - T_1\left(e_1, \alpha_1^0\rho_1^0/\alpha_1\right) = 0$$

pT relaxation: Algebraic approach

As before, for $k = 1, 2$, states remain in equilibrium are

$$\alpha_k \rho_k = \alpha_{k0} \rho_{k0}, \quad \rho = \rho_0, \quad \vec{u} = \vec{u}_0, \quad \rho E = (\rho E)_0, \quad e = e_0$$

Lead to equilibrium in mass fraction $Y_k = \alpha_k \rho_k / \rho = Y_{k0}$

pT relaxation: Algebraic approach

As before, for $k = 1, 2$, states remain in equilibrium are

$$\alpha_k \rho_k = \alpha_{k0} \rho_{k0}, \quad \rho = \rho_0, \quad \vec{u} = \vec{u}_0, \quad \rho E = (\rho E)_0, \quad e = e_0$$

Lead to equilibrium in mass fraction $Y_k = \alpha_k \rho_k / \rho = Y_{k0}$

Impose mechanical-thermal equilibrium to

1. Saturation condition

$$\frac{\alpha_1 \rho_1}{\rho_1(p, T)} + \frac{\alpha_2 \rho_2}{\rho_2(p, T)} = 1$$

   or

$$\frac{Y_1}{\rho_1(p, T)} + \frac{Y_2}{\rho_2(p, T)} = \frac{1}{\rho}$$

Impose mechanical-thermal equilibrium to

1. Saturation condition

$$\frac{Y1}{\rho_1(p,T)} + \frac{Y_2}{\rho_2(p,T)} = \frac{1}{\rho}$$

2. Equilibrium of internal energy

$$Y_1\, e_1(p,T) + Y_2\, e_2(p,T) = e$$

Give 2 algebraic equations for 2 unknowns p & T

Impose mechanical-thermal equilibrium to

1. Saturation condition

$$\frac{Y1}{\rho_1(p,T)} + \frac{Y_2}{\rho_2(p,T)} = \frac{1}{\rho}$$

2. Equilibrium of internal energy

$$Y_1\, e_1(p,T) + Y_2\, e_2(p,T) = e$$

Give 2 algebraic equations for 2 unknowns p & T

For SG EOS, it reduces to single quadratic equation for p & explicit computation of T:

$$\frac{1}{\rho T} = Y_1\frac{(\gamma_1 - 1)C_{v1}}{p + p_{\infty 1}} + Y_2\frac{(\gamma_2 - 1)C_{v2}}{p + p_{\infty 2}}$$

pTg relaxation

Look for solution of ODEs in limits $\mu,\ \theta,\ \&\ \nu \to \infty$

$$\partial_t \left(\alpha_1 \rho_1\right) = \nu \left(g_2 - g_1\right)$$
$$\partial_t \left(\alpha_2 \rho_2\right) = \nu \left(g_1 - g_2\right)$$
$$\partial_t \left(\rho \vec{u}\right) = 0$$
$$\partial_t \left(\alpha_1 \rho_1 E_1\right) = \mu p_I \left(p_2 - p_1\right) + \theta T_I \left(T_2 - T_1\right) + \nu \left(g_2 - g_1\right)$$
$$\partial_t \left(\alpha_2 \rho_2 E_2\right) = \mu p_I \left(p_1 - p_2\right) + \theta T_I \left(T_1 - T_2\right) + \nu \left(g_1 - g_2\right)$$
$$\partial_t \alpha_1 = \mu \left(p_1 - p_2\right) + \nu v_I \left(g_2 - g_1\right)$$

under mechanical-thermal-chemical equilibrium conditons

$$p_1 = p_2 = p$$
$$T_1 = T_2 = T$$
$$g_1 = g_2$$

pTg relaxation: Algebraic apporach

In this case, states remain in equilibrium are

$$\rho = \rho_0, \quad \rho\vec{u} = \rho_0\vec{u}_0, \quad \rho E = (\rho E)_0, \quad e = e_0$$

but $\alpha_k \rho_k \neq \alpha_{k0}\rho_{k0}$ & $Y_k \neq Y_{k0}$, $k = 1, 2$

Impose mechanical-thermal-chemical equilibrium to
1. Saturation condition for temperature
$$\mathcal{G}(p, T) = 0$$

2. Saturation condition for volume fraction
$$\frac{Y_1}{\rho_1(p, T)} + \frac{Y_2}{\rho_2(p, T)} = \frac{1}{\rho}$$

3. Equilibrium of internal energy
$$Y_1\, e_1(p, T) + Y_2\, e_2(p, T) = e$$

From saturation condition for temperature

$$\mathcal{G}(p, T) = 0$$

we get T in terms of p, while from

$$\frac{Y_1}{\rho_1(p, T)} + \frac{Y_2}{\rho_2(p, T)} = \frac{1}{\rho}$$

&

$$Y_1\, e_1(p, T) + Y_2\, e_2(p, T) = e$$

we obtain algebraic equation for p

$$Y_1 = \frac{1/\rho_2(p) - 1/\rho}{1/\rho_2(p) - 1/\rho_1(p)} = \frac{e - e_2(p)}{e_1(p) - e_2(p)}$$

which is solved by iterative method

- Having known Y_k & p, T can be solved from, *e.g.*,

$$Y_1 \, e_1(p, T) + Y_2 \, e_2(p, T) = e^0$$

  yielding update ρ_k & α_k

- Feasibility of solutions, *i.e.*, positivity of physical quantities ρ_k, α_k, p, & T, for example
  - Employ hybrid method *i.e.*, combination of above method with differential-based approach (not discuss here), when it becomes necessary

Dodecane 2-phase Riemann problem

Comparison p-,pT-& p-pTg-relaxation solution at $t = 473\mu$s
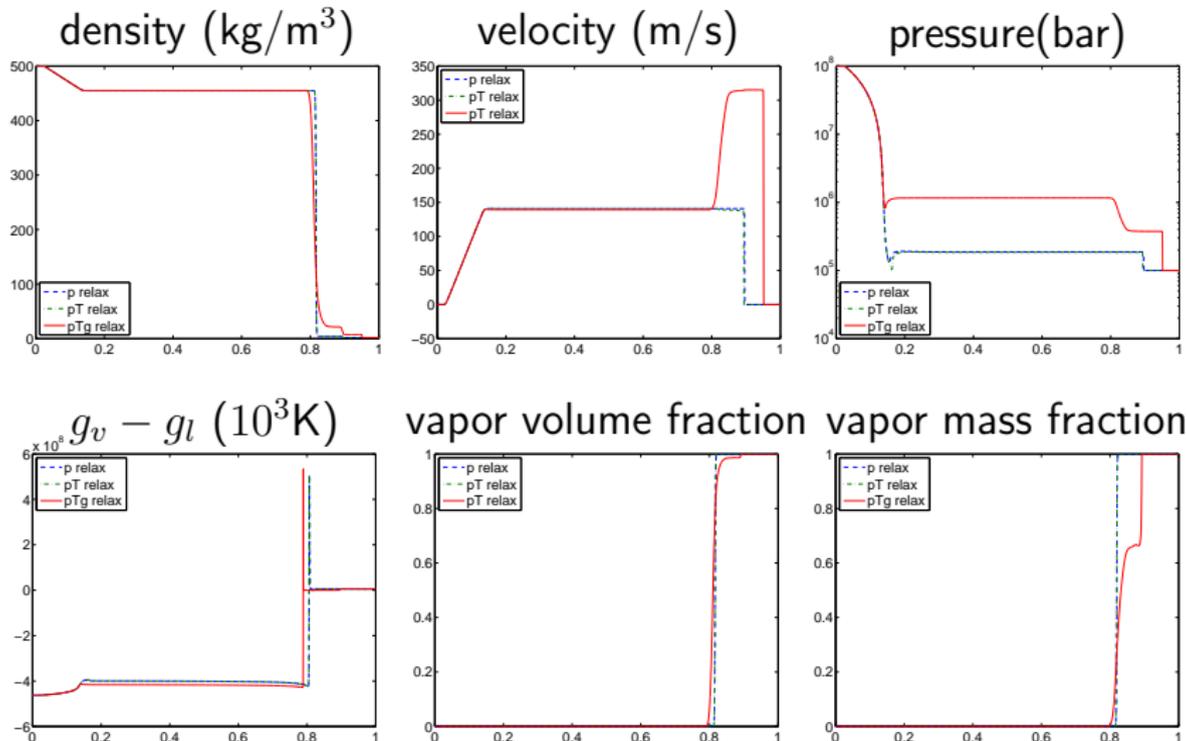
density (kg/m^3)

velocity (m/s)

pressure(bar)

$g_v - g_l$ (10^3K)

vapor volume fraction

vapor mass fraction
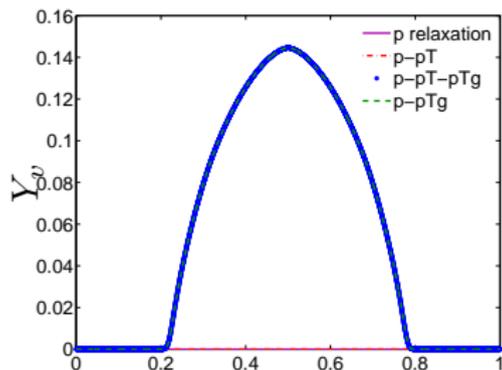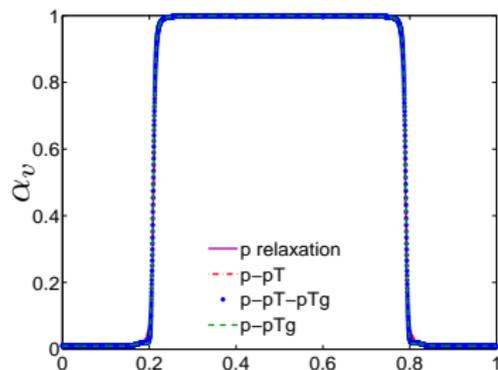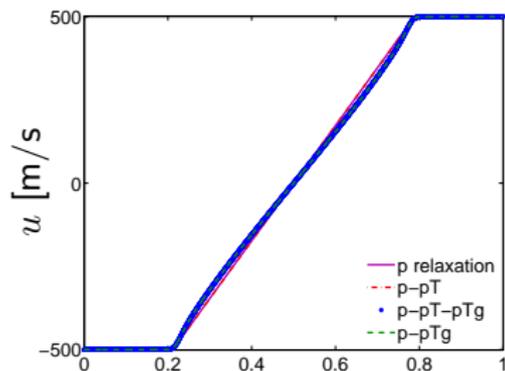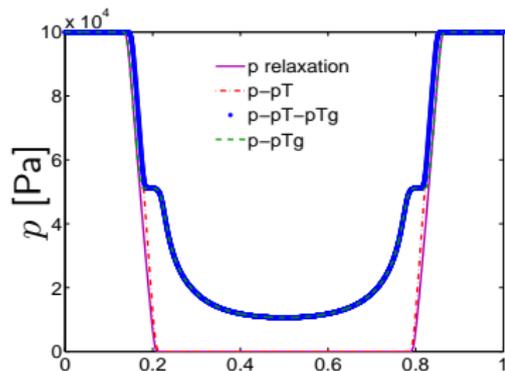
Consider solution of model system

$$\partial_t w + \nabla \cdot f(w) + \mathcal{B}(w, \nabla w) = 0$$

in 2D general non-rectangular geometry
Model in integral form over any control volume C is

$$\frac{d}{dt} \int_C w \, d\Omega = - \int_{\partial C} f(w) \cdot \vec{n} \, ds - \int_C \mathcal{B}(w, \nabla w) \, d\Omega$$

where $\vec{n}$ is outward-pointing normal vector at boundary ∂C
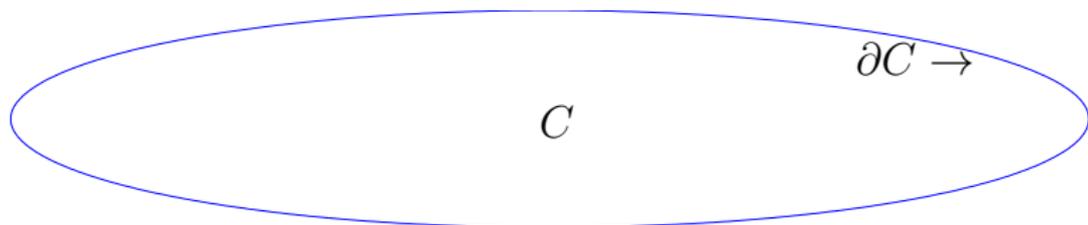
Then finite volume method on control volume C reads

$$W^{n+1} = W^n - \frac{\Delta t}{\mathcal{M}(C)} \sum_{j=1}^{N_s} h_j \breve{F}_j - \Delta t \mathcal{B}^* \mathcal{M}(C)$$

- $\mathcal{M}(C)$ measure (area in 2D or volume in 3D) of C
- N_s number of sides
- h_j length of j-th side (in 2D) or area of cell edge (in 3D) measured in physical space
- $\breve{F}_j$ numerical approximation to normal flux in average across j-th side of grid cell
- $\mathcal{B}^* := \int_C \mathcal{B}(z, t_n)\, dz / \mathcal{M}(C)$ (cell average of $\mathcal{B}$ in cell C)

Assume mapped (*i.e.*, logically rectangular) grid in 2D, we get

$$W_{ij}^{n+1} = W_{ij}^n - \frac{\Delta t}{\kappa_{ij}\Delta\xi_1}\left(F_{i+\frac{1}{2},j}^1 - F_{i-\frac{1}{2},j}^1\right) - $$

$$\frac{\Delta t}{\kappa_{ij}\Delta\xi_2}\left(F_{i,j+\frac{1}{2}}^2 - F_{i,j-\frac{1}{2}}^2\right) - \Delta t \mathcal{B}_{ij}^* \Delta\xi_1 \Delta\xi_2$$


physical grid

logical grid
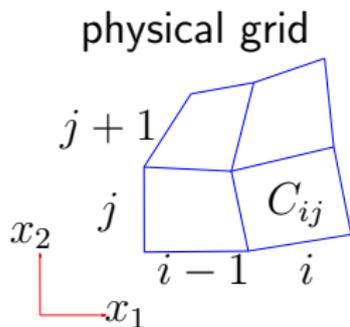
mapping
←

$x_1 = x_1(\xi_1, \xi_2)$
$x_2 = x_2(\xi_1, \xi_2)$

$\kappa_{ij} = \mathcal{M}(C_{ij})/\Delta\xi_1\Delta\xi_2$

Godunov-type in wave propagation form for hyperbolic conservation law is

$$W_{ij}^{n+1} = W_{ij}^n - \frac{\Delta t}{\kappa_{ij}\Delta\xi_1}\left(\mathcal{A}_1^+\Delta W_{i-\frac{1}{2},j} + \mathcal{A}_1^-\Delta W_{i+\frac{1}{2},j}\right) -$$

$$\frac{\Delta t}{\kappa_{ij}\Delta\xi_2}\left(\mathcal{A}_2^+\Delta W_{i,j-\frac{1}{2}} + \mathcal{A}_2^-\Delta W_{i,j+\frac{1}{2}}\right)$$

- fluctuations $\mathcal{A}_1^+\Delta W_{i-\frac{1}{2},j}$, $\mathcal{A}_1^-\Delta W_{i+\frac{1}{2},j}$, $\mathcal{A}_2^+\Delta W_{i,j-\frac{1}{2}}$, & $\mathcal{A}_2^-\Delta W_{i,j+\frac{1}{2}}$: Solve one-dimensional Riemann problems in direction normal to cell edges

- $\mathcal{B}_{ij}^*$ may be included in fluctuations
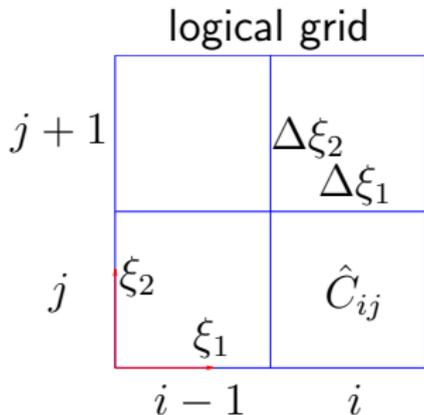
Speeds & limited of waves are used to calculate second order correction:

$$W_{ij}^{n+1} := W_{ij}^{n+1} - \frac{\Delta t}{\kappa_{ij}\Delta\xi_1}\left(\widetilde{\mathcal{F}}_{i+\frac{1}{2},j}^1 - \widetilde{\mathcal{F}}_{i-\frac{1}{2},j}^1\right) - $$
$$\frac{\Delta t}{\kappa_{ij}\Delta\xi_2}\left(\widetilde{\mathcal{F}}_{i,j+\frac{1}{2}}^2 - \widetilde{\mathcal{F}}_{i,j-\frac{1}{2}}^2\right)$$

For example, at cell edge $(i-\frac{1}{2},j)$ correction flux takes

$$\widetilde{\mathcal{F}}_{i-\frac{1}{2},j}^1 = \frac{1}{2}\sum_{k=1}^{N_w}\left|\lambda_{i-\frac{1}{2},j}^{1,k}\right|\left(1 - \frac{\Delta t}{\kappa_{i-\frac{1}{2},j}\Delta\xi_1}\left|\lambda_{i-\frac{1}{2},j}^{1,k}\right|\right)\widetilde{\mathcal{W}}_{i-\frac{1}{2},j}^{1,k}$$

$\kappa_{i-\frac{1}{2},j} = (\kappa_{i-1,j} + \kappa_{ij})/2$, $\widetilde{\mathcal{W}}_{i-\frac{1}{2},j}^{1,k}$ is limited waves to avoid oscillations near discontinuities

Transverse wave propagation is included to ensure second order accuracy & also improve stability
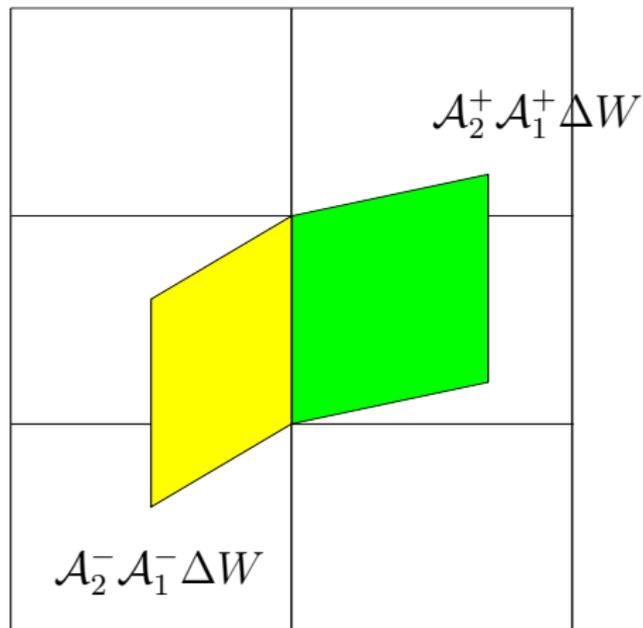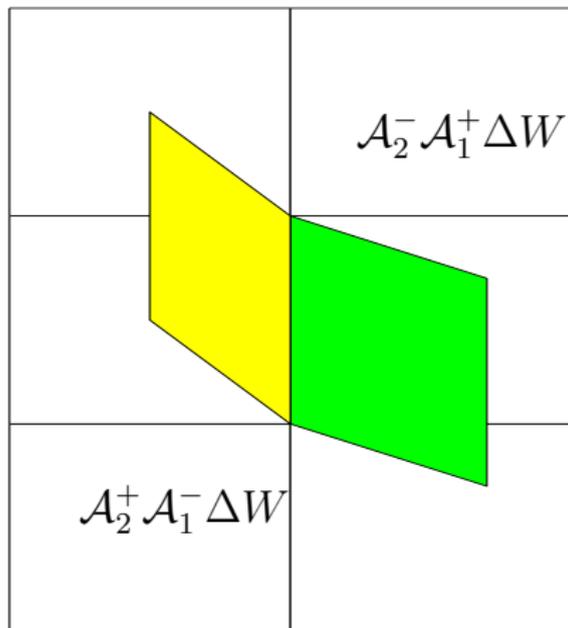
Method can be shown to be quasi conservative & stable under a variant of CFL (Courant-Friedrichs-Lewy) condition

$$\Delta t \max_{i,j,k} \left(\frac{\left| \lambda^{1,k}_{i-\frac{1}{2},j} \right|}{\kappa_{i_p,j} \Delta \xi_1}, \frac{\left| \lambda^{2,k}_{i,j-\frac{1}{2}} \right|}{\kappa_{i,j_p} \Delta \xi_2} \right) \leq 1,$$

$$i_p = i \quad \text{if } \lambda^{1,k}_{i-\frac{1}{2},j} > 0 \quad \& \quad i-1 \quad \text{if } \lambda^{1,k}_{i-\frac{1}{2},j} < 0$$

Semi-discrete wave propagation method takes form

$$\partial_t W(t) = \mathcal{L}(W(t))$$

where in 2D

$$\mathcal{L}(W_{ij}(t) = -\frac{1}{\kappa_{ij}\Delta\xi_1}\left(\mathcal{A}_1^+\Delta W_{i-\frac{1}{2},j} + \mathcal{A}_1^-\Delta W_{i+\frac{1}{2},j} + \mathcal{A}_1\Delta W_{ij}\right) -$$

$$\frac{1}{\kappa_{ij}\Delta\xi_2}\left(\mathcal{A}_2^+\Delta W_{i,j-\frac{1}{2}} + \mathcal{A}_2^-\Delta W_{i,j+\frac{1}{2}} + \mathcal{A}_2\Delta W_{ij}\right)$$

ODEs are integrated in time using strong stability-preserving (SSP) multistage Runge-Kutta, *e.g.*, 3-stage 3rd-order

$$W^* = W^n + \Delta t\mathcal{L}\left(W^n\right)$$

$$W^{**} = \frac{3}{4}W^n + \frac{1}{4}W^* + \frac{1}{4}\Delta t\mathcal{L}\left(W^*\right)$$

$$W^{n+1} = \frac{1}{3}W^n + \frac{2}{3}W^* + \frac{2}{3}\Delta t\mathcal{L}\left(W^{**}\right)$$