Homogeneous relaxation models & methods for compressible three-phase flow & more

II: Future perspectives

Keh-Ming Shyue

Institute of Applied Mathematical Sciences
National Taiwan University

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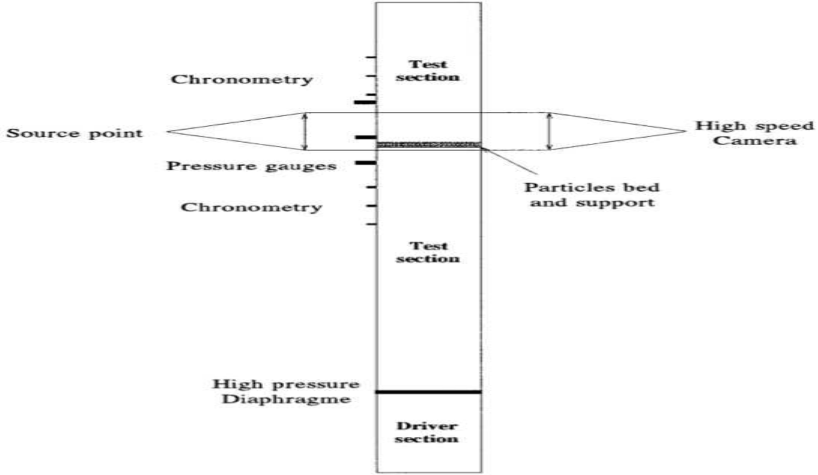
Shock-induced particle fluidization: setup

Schematic experimental setup (taken from Rogue *et al.* (Shock wave 1998) & so plots in following 4 slides)



Particle fluidization: Drag coefficient measurement

Table 1. Summary of experimental results on drag coefficient

Gases Driver/Driven	Mach number	Particle diameter (mm)	Particle material	C_d	Reynolds range
Air/Air	1.291	2	Glass	0.61	27 000–30 000
Air/Air	1.298	2	Nylon	0.57	25 000–31 000
Air/Air	1.29	1	Glass	0.54	12 000–14 800
SF_6/SF_6	1.341	2	Glass	0.62	80 000–119 000
SF_6/SF_6	1.341	2	Nylon	0.62	79 000–116 000
SF_6/SF_6	1.354	1	Glass	0.58	44 000–62 000
SF_6/Air	1.217	2	Nylon	0.51	17 000–21 000
SF_6/Air	1.218	1	Glass	0.50	7200–10 000
Air/Helium	1.181	2	Nylon	0.57	5000–6000

Particle fluidization: Collective particle behavior

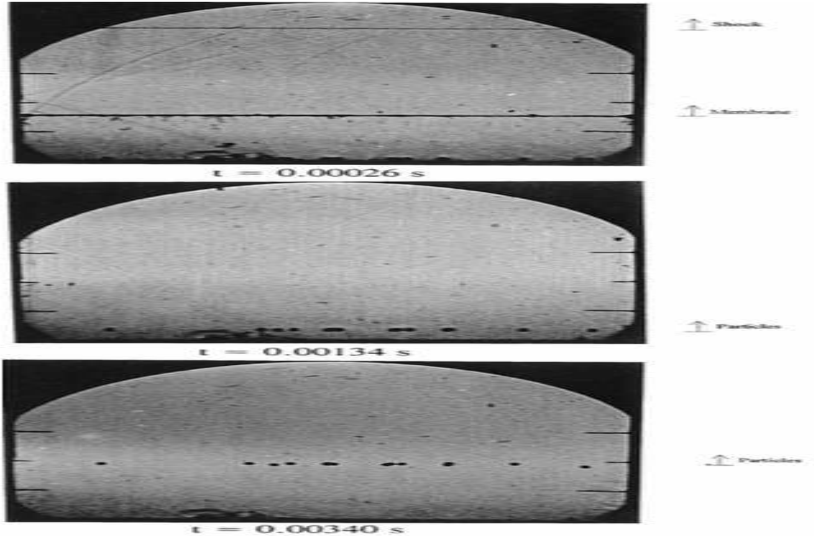


Fig. 7. A few 2 mm diameter nylon particles on a plastic membrane, shock Mach= 1.3 in air

Particle fluidization: Collective particle behavior

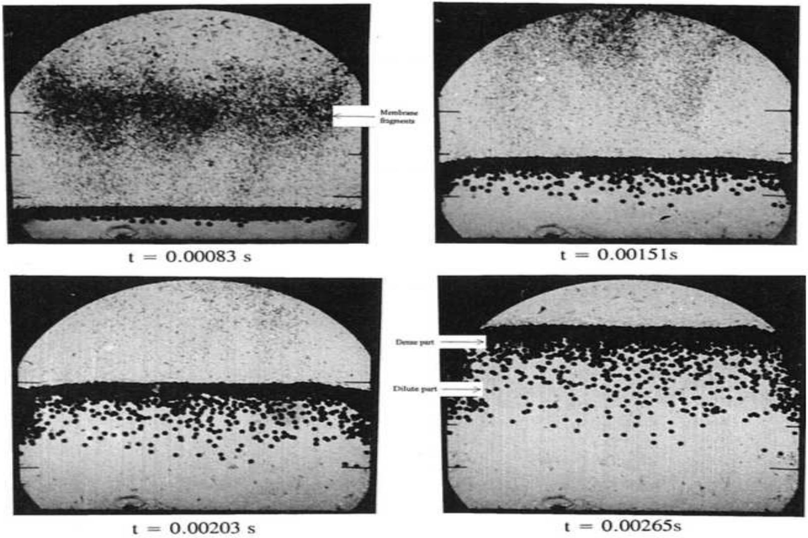


Fig. 9. A single layer of 2 mm diameter glass particles on a plastic membrane, Shock Mach=1.3 in air

Particle fluidization: Collective particle behavior

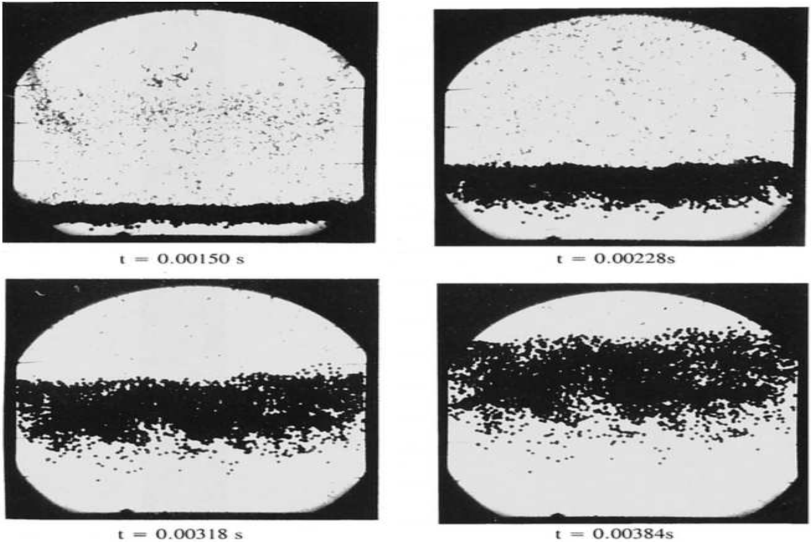


Fig. 11. A double layer of 2 mm diameter glass particles on a plastic membrane, Shock Mach=1.3 in air

Particle fluidization: Collective particle behavior

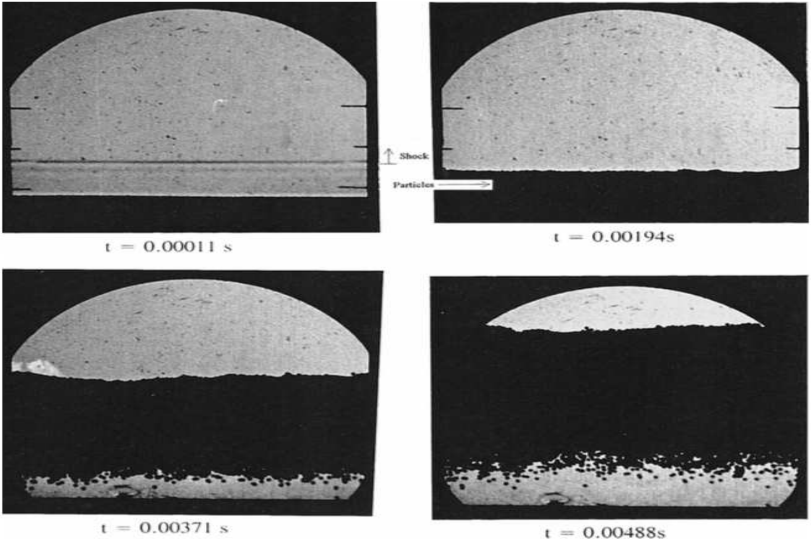
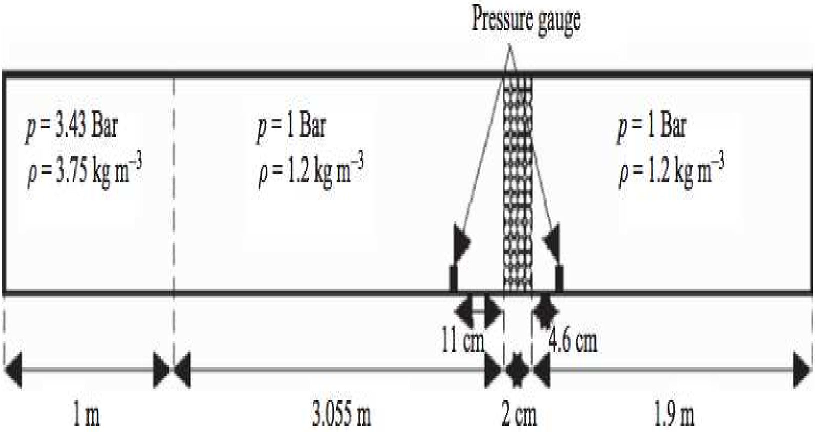


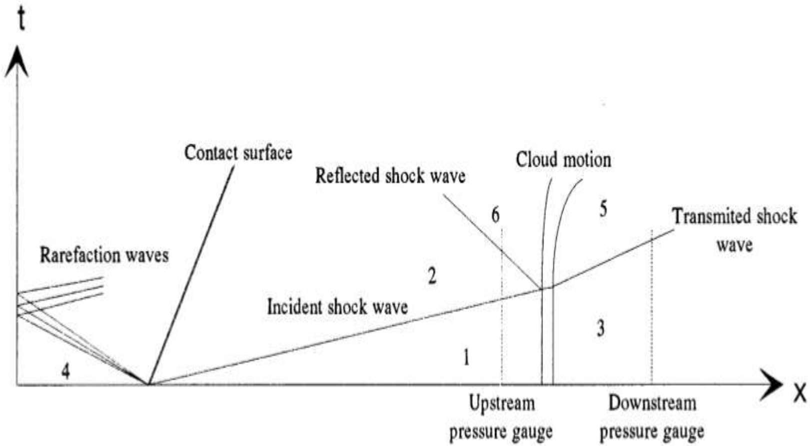
Fig. 13. A 2 cm thick bed of 1.5 mm diameter glass particles on a metal grid, shock Mach=1.3 in air

Particle fluidization: numerical setup



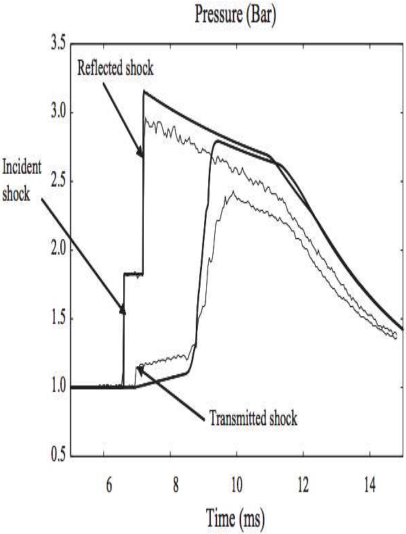
Particle fluidization: Wave pattern

Schematic x-t diagram of two-phase flow in shock tube

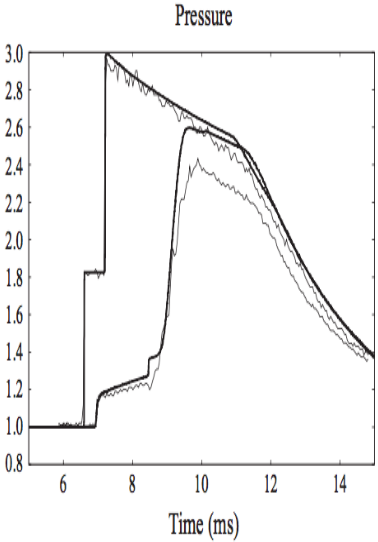


Particle fluidization: Diagnosis

without drift velocity

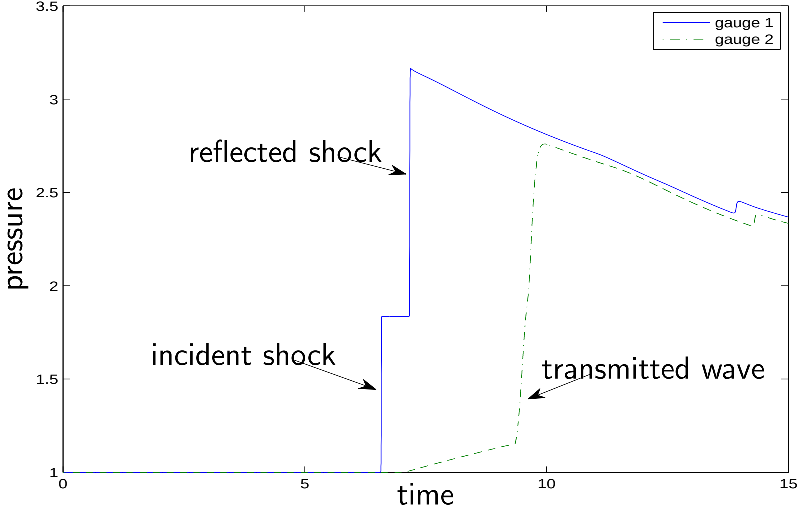


with drift velocity



Particle fluidization: Diagnosis

Results obtained using 5-equation transport model



Particle fluidization: 6-equation model

Saurel *et al.* (JFM 2010) proposed compressible 2-phase (fluid-solid (granular material)) model as

$$
\begin{aligned}
&\partial_t\left(\alpha_1\rho_1\right) + \nabla\cdot\left(\alpha_1\rho_1\vec{u}\right) = 0\\
&\partial_t\left(\alpha_2\rho_2\right) + \nabla\cdot\left(\alpha_2\rho_2\vec{u}\right) = 0\\
&\partial_t\left(\rho\vec{u}\right) + \nabla\cdot\left(\rho\vec{u}\otimes\vec{u}\right) + \nabla p = 0\\
&\partial_t\left((\alpha\rho)_1(e+B)_1\right) + \nabla\cdot\left((\alpha\rho)_1(e+B)_1\vec{u}\right) + (\alpha p)_1\,\nabla\cdot\vec{u} =\\
&\qquad\qquad\mu\pi_I(\pi_2-\pi_1)\\
&\partial_t\left((\alpha\rho)_2(e+B)_2\right) + \nabla\cdot\left((\alpha\rho)_2(e+B)_2\vec{u}\right) + (\alpha p)_2\,\nabla\cdot\vec{u} =\\
&\qquad\qquad-\,\mu\pi_I(\pi_2-\pi_1)\\
&\partial_t(\rho E) + \nabla\cdot\left(\rho E\vec{u} + p\vec{u}\right) = 0\\
&\partial_t\alpha_1 + \vec{u}\cdot\nabla\alpha_1 = \mu(\pi_1-\pi_2)
\end{aligned}
$$

π_k: phasic total stress, β_k: granular pressure, & B_k granular energy; $k=1$ solid phase & $k=2$ fluid phase

In addition, for $k = 1, 2$, we have

$$
\begin{aligned}
\pi_k &= p_k - \beta_k \\
\pi_I &= \left(\rho_2 c_2 \pi_1 + \rho_1 c_1 \pi_2\right) / \left(\rho_1 c_1 + \rho_2 c_2\right) \\
E_k &= e_k + B_k + \frac{1}{2}\vec{u} \cdot \vec{u} \\
E &= Y_1 E_1 + Y_2 E_2 \\
p &= \alpha_1 p_1 + \alpha_2 p_2
\end{aligned}
$$

Relaxation parameter μ assumed to satisfy

$$
\mu = \begin{cases} \infty & \text{if } \pi_1 > \pi_2, \\ 0 & \text{otherwise,} \end{cases}
$$

Constitutive laws

1. Equation of state for fluid (stiffened gas EOS)

$$p_k = (\gamma_k - 1)\rho_k e_k - \gamma_k p_{\infty,k}$$

2. Granular energy of the form

$$B_k(\alpha) = \begin{cases} B_a(\alpha) & \text{if } \alpha_0 < \alpha < 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$B_a(\alpha) = a\left[(1-\alpha)\log(1-\alpha) + \right.$$
$$\left.(1+\log(1-\alpha_0))(\alpha-\alpha_0) - (1-\alpha_0)\log(1-\alpha_0)\right]^n$$

3. Granular pressure

$$\beta_k = (\alpha\rho)_k \frac{dB_k}{d\alpha_k}$$

Parameters a, n, & α_0 are material-dependent quantities for given powder

7-equation model

Baer-Nunziato (Intl. J. Multiphase Flows 1986)

$$
\begin{aligned}
&\partial_t \left(\alpha_1 \rho_1\right) + \nabla \cdot \left(\alpha_1 \rho_1 \vec{u}_1\right) = 0 \\
&\partial_t \left(\alpha_2 \rho_2\right) + \nabla \cdot \left(\alpha_2 \rho_2 \vec{u}_2\right) = 0 \\
&\partial_t(\alpha_1 \rho_1 \vec{u}_1) + \nabla \cdot \left(\alpha_1 \rho_1 \vec{u}_1 \otimes \vec{u}_1\right) + \nabla \left(\alpha_1 p_1\right) = \pi_I \nabla \alpha_1 - \lambda \vec{u}_R \\
&\partial_t(\alpha_2 \rho_2 \vec{u}_2) + \nabla \cdot \left(\alpha_2 \rho_2 \vec{u}_2 \otimes \vec{u}_2\right) + \nabla \left(\alpha_2 p_2\right) = -\pi_I \nabla \alpha_1 + \lambda \vec{u}_R \\
&\partial_t \left(\alpha_1 \rho_1 E_1\right) + \nabla \cdot \left(\alpha_1 \rho_1 E_1 \vec{u}_1 + \alpha_1 p_1 \vec{u}_1\right) = \\
&\qquad\qquad \pi_I \vec{u}_I \nabla \alpha_1 - \lambda \vec{u}_I' \vec{u}_R - \mu \pi_I \pi_R \\
&\partial_t \left(\alpha_2 \rho_2 E_2\right) + \nabla \cdot \left(\alpha_2 \rho_2 E_2 \vec{u}_2 + \alpha_2 p_2 \vec{u}_2\right) = \\
&\qquad\qquad - \pi_I \vec{u}_I \nabla \alpha_1 + \lambda \vec{u}_I' \vec{u}_R + \mu \pi_I \pi_R \\
&\partial_t \alpha_1 + \vec{u} \cdot \nabla \alpha_1 = \mu \pi_R
\end{aligned}
$$

$$
\pi_R = \pi_1 - \pi_2 \quad \& \quad \vec{u}_R = \vec{u}_1 - \vec{u}_2
$$

μ & λ: Relaxation parameters

5-equation model with drift effects

Saurel *et al.* (JFM 2010)

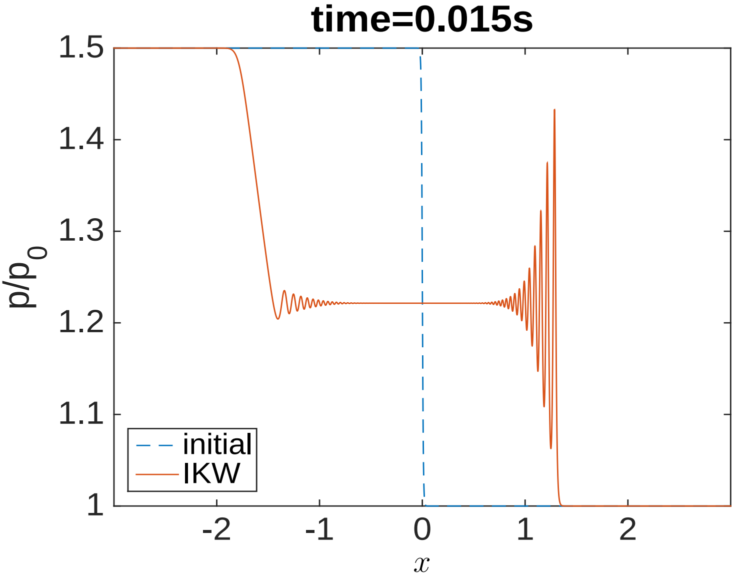
$$
\begin{aligned}
&\partial_t\left(\alpha_1\rho_1\right)+\nabla\cdot\left(\alpha_1\rho_1\vec{u}\right)+\nabla\cdot\left(\rho Y_1Y_2\vec{u}_R\right)=0\\
&\partial_t\left(\alpha_2\rho_2\right)+\nabla\cdot\left(\alpha_2\rho_2\vec{u}\right)-\nabla\cdot\left(\rho Y_1Y_2\vec{u}_R\right)=0\\
&\partial_t(\rho\vec{u})+\nabla\cdot(\rho\vec{u}\otimes\vec{u})+\nabla p=0\\
&\partial_t\left(\rho E\right)+\nabla\cdot\left(\rho E\vec{u}+p\vec{u}\right)+\\
&\qquad\qquad\nabla\cdot\left(\rho Y_1Y_2\vec{u}_R((h+B)_1-(h+B)_2)\right)=0\\
&\partial_t\alpha_1+\vec{u}\cdot\nabla\alpha_1=\\
&\qquad\rho c_w^2\frac{\alpha_1\alpha_2}{\rho_1c_1^2\rho_2c_2^2}\left[((\rho_2c_2^2-\beta_2)-(\rho_1c_1^2-\beta_1))\nabla\cdot\vec{u}-\right.\\
&\qquad\qquad\left(\frac{\rho_1c_1^2-\beta_1}{\alpha_1\rho_1}+\frac{\rho_2c_2^2-\beta_2}{\alpha_2\rho_2}\right)\nabla\cdot\left(\rho Y_1Y_2\vec{u}_R\right)-\\
&\qquad\qquad\left.\rho Y_1Y_2\vec{u}_R\cdot\left(\frac{\Gamma_1}{\alpha_1}T_1\nabla s_1+\frac{\Gamma_2}{\alpha_2}T_2\nabla s_2\right)\right]
\end{aligned}
$$

$u_R \approx \vec{u}_1 - \vec{u}_2$ takes form

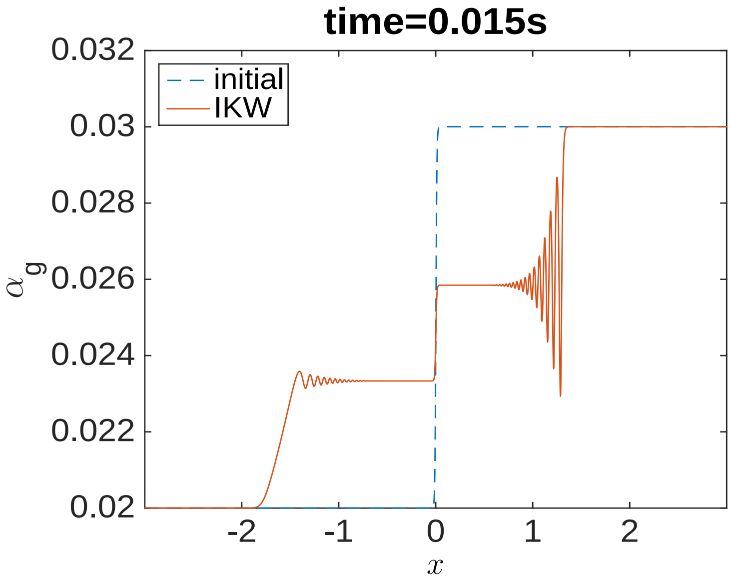
$$\vec{u}_R = \frac{1}{\lambda}\left(\left(\alpha_1 - Y_1\right)\nabla p - \alpha_2 \nabla\left(\alpha_1 \beta_1\right) + \alpha_1 \nabla\left(\alpha_2 \beta_2\right)\right)$$

There exists 6-equation model with drift effects (not given here)

Riemann problem: IKW model



Riemann problem: IKW model



Bubbly flow in liquid

Iordanski-Kogarko-Wijngaarden model for bubbly flow in liquid takes form

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0 \\
&\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla p = 0 \\
&\partial_t (\alpha_g \rho_g) + \nabla \cdot (\alpha_g \rho_g \vec{u}) = 0 \\
&\partial_t N_b + \nabla \cdot (N_b \vec{u}) = 0
\end{aligned}$$

To close the model, assumptions are:

1. The liquid phase is assumed to be incompressible, where ρ_l is set to be a chosen constant.
2. The volume fraction of the gas α_g is assumed to satisfy the relation
$$\alpha_g = \frac{4}{3}\pi R^3 N_b,$$
where R is the bubble radius.

3. The gas inside the bubble is assumed to be ideal & takes

$$p_g = p_0 \left(\frac{R_0}{R}\right)^{3\gamma},$$

4. Assume mixture pressure p follow Rayleigh-Plesset equation

$$p = p_g(R) - \rho_l \left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right) - 4\mu_l \frac{\dot{R}}{R}.$$

   Here μ_l is the dynamic viscosity of the liquid, and $\dot{R}$ means dR/dt.

5. Bubble distribution is assumed to be uniform, *i.e.*, the flow is in the absence of bubble breakup and coalescence

Impact-shear 2-phase Riemann problem

Table: Material-dependent quantities used in the stiffened gas equation of state for the simulation

	γ	p_∞(GPa)	μ(GPa)	ρ_0(kg/m^3)
Titanium	2.6	44	42	4527
Aluminum	3.5	3.2	26	2712

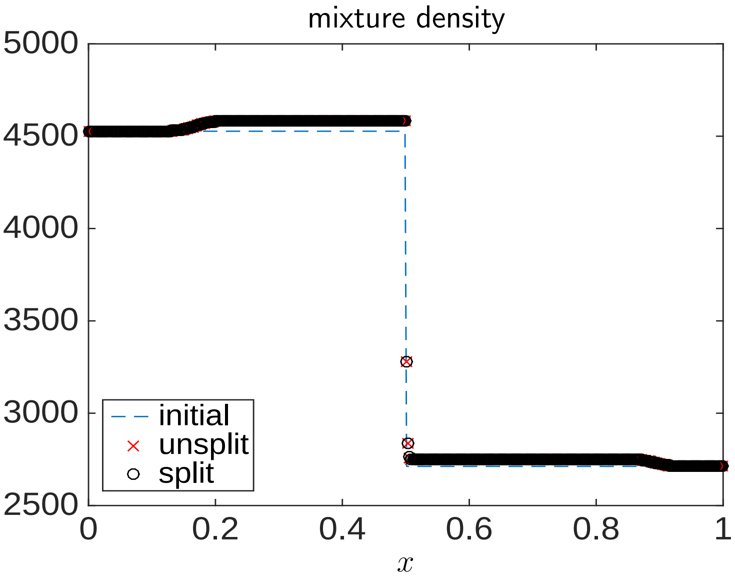
Initial condition:

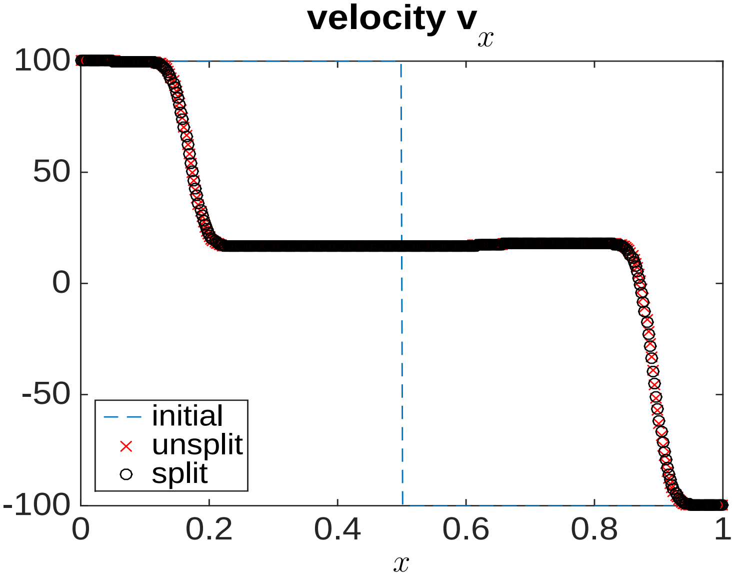
- Titanum phase: Left-hand side ($0 \leq x \leq 0.5$m)

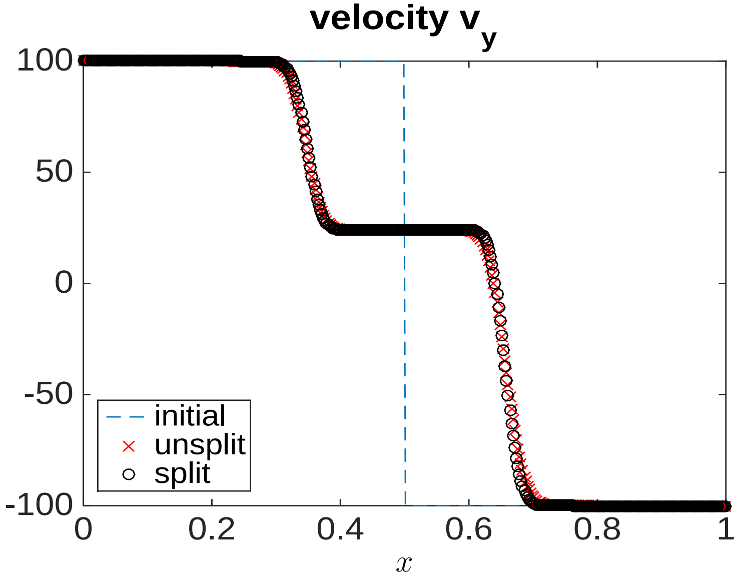
$$(\rho_1, \rho_2, u, v, p, \alpha_1)_L = \left(4527, 2712, 10^2, 10^2, 10^5, 1 - 10^{-8}\right)$$

- Aluminum phase: Right-hand side (0.5m $< x \leq 1$m)

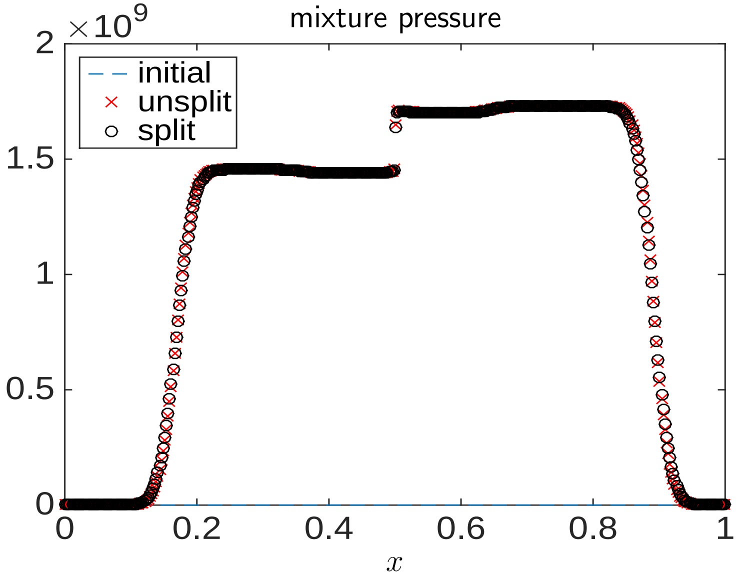
$$(\rho_1, \rho_2, u, v, p, \alpha_1)_L = \left(4527, 2712, -10^2, -10^2, 10^5, 10^{-8}\right)$$


mixture density
5000
4500
4000
3500
3000
2500
0
0.2
0.4
0.6
0.8
1
x
initial
unsplit
split

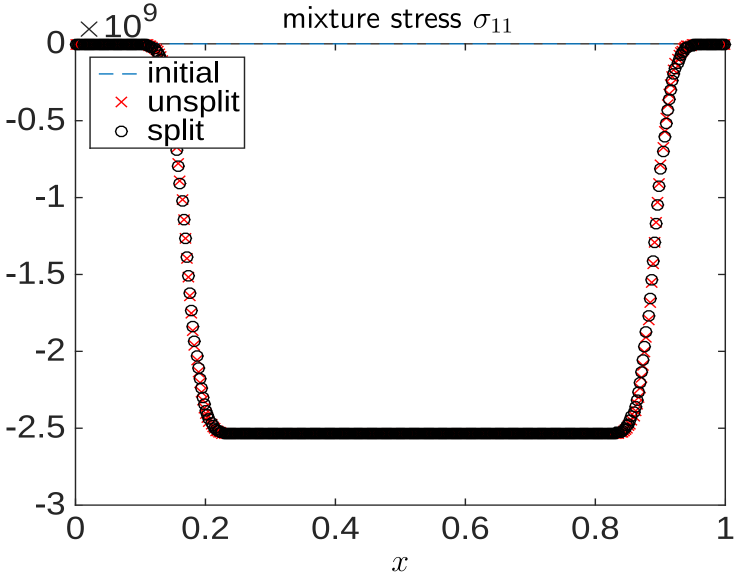

velocity v$_x$
100
50
0
-50
-100
0
0.2
0.4
0.6
0.8
1
x
initial
unsplit
split



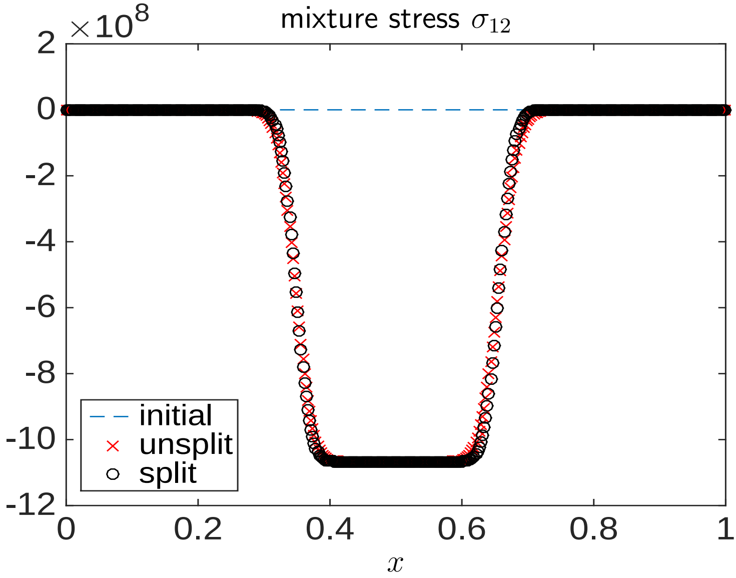
velocity v_y
100
50
0
-50
-100
0
0.2
0.4
0.6
0.8
1
x
initial
unsplit
split



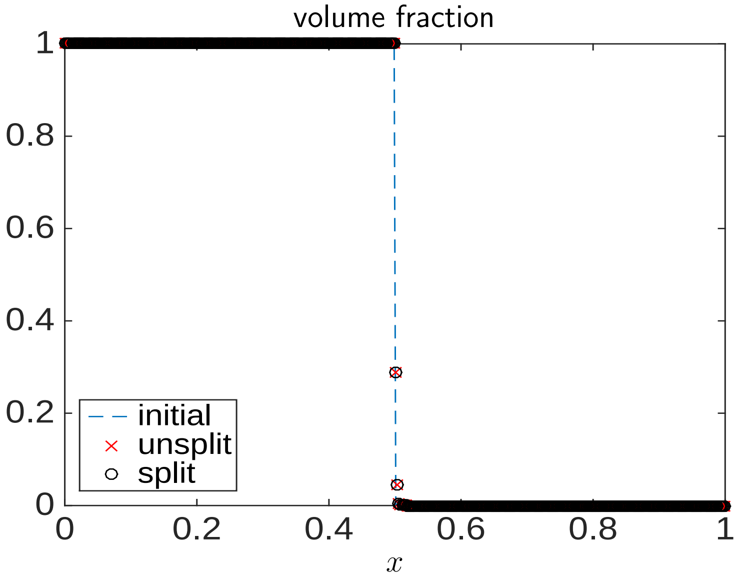
mixture pressure
×10^9
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
x
initial
unsplit
split



mixture stress σ_{11}
$\times 10^9$
0
-0.5
-1
-1.5
-2
-2.5
-3
0
0.2
0.4
0.6
0.8
1
x
initial
unsplit
split



mixture stress σ_{12}
$\times 10^8$
2
0
-2
-4
-6
-8
-10
-12
0
0.2
0.4
0.6
0.8
1
x
initial
unsplit
split


volume fraction
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
x
initial
unsplit
split

Model for hyperelastic flow

Ndanou *et al.* (JCP 2015) proposed

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$
$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \otimes u - \sigma) = 0$$
$$\frac{\partial E}{\partial t} + \nabla \cdot (Eu - \sigma u) = 0$$
$$\frac{\partial e^\beta}{\partial t} + \nabla \left(u \cdot e^\beta\right) = 0$$
$$\nabla \times e^\beta = 0$$

Consider internal energy e_k for each phase in separable form

$$e_k = e_k^h(\rho_k, p_k) + e_k^e(g_k)$$

Assume hydrodynamic energy e_k^h follows stiffened gas EOS as

$$e_k^h\left(\rho_k, p_k\right) = \frac{p_k + \gamma_k p_{\infty k}}{\gamma_k - 1}$$

For isotropic solids, elastic energy e_k^e assumes

$$\begin{aligned} e_k^e(g_k) &= e_k^e\left(j_{1k}, j_{2k}\right) \\ j_{1k} &= \mathsf{trace}\left(g_k\right) \\ j_{2k} &= \mathsf{trace}\left(g_k^2\right) \end{aligned}$$

For applications, we take

$$e_k^e(g_k) = \frac{\mu_k}{8\rho_{0k}}\left(j_{2k} - 3\right)$$

or

$$e_k^e(g_k) = \frac{\mu_k}{4\rho_{0k}}\left(\theta j_{2k} + \frac{1-2\theta}{3} j_{1k}^2 + 3(\theta - 1)\right)$$

$$g_k = \frac{G_k}{|G_k|^{1/3}}, \quad |G_k| = \mathsf{det}(G_k)$$

Let $X_k = \{X_k^i\}$ be Lagrangian coordinates for solid k

Denote e_k^i as being local cobasis of form

$$e_k^i = \nabla_x X_k^i = \left(a_k^i,\ b_k^i,\ c_k^i\right)$$

Define deformation gradient F_k as

$$F_k^{-T} = \left(e_k^1,\ e_k^2,\ e_k^3\right)$$

Then Finger tensor G_k takes form

$$G_k = F_k^{-T} F_k^{-1} = \sum_{i=1}^{3} e_k^i \otimes e_k^i$$

Denote σ to be total stress as

$$\sigma = \sum_k \alpha_k \sigma_k$$

with phasic stress tensor σ_k defined by

$$\begin{aligned}
\sigma_k &= -2\rho_k \frac{\partial e_k}{\partial G_k} G_k = -p_k I + S_k \\
p_k &= \rho_k^2 \frac{\partial e_k^h}{\partial \rho_k} \\
S_k &= -2\rho_k \frac{\partial e_k^e}{\partial G_k} G_k \\
&= -2\rho_k \left[2 \frac{\partial e_k^e}{\partial j_{2k}} \left(g_k^2 - \frac{j_{2k}}{3} I \right) + \frac{\partial e_k^e}{\partial j_{1k}} \left(g_k - \frac{j_{1k}}{3} I \right) \right]
\end{aligned}$$

Model is hyperbolic with sound speed $c \in \mathbf{R}$ defined by

$$
\begin{aligned}
c_k^2 = \frac{\partial p}{\partial \rho} - \frac{\partial S_{11}}{\partial \rho} - \frac{1}{\rho}\left(\frac{\partial S_{11}}{\partial a} \cdot a\right) \\
- \frac{\partial S_{11}}{\partial \rho} - \frac{1}{\rho}\left(\frac{\partial S_{11}}{\partial a} \cdot a\right) = \frac{\partial e^e}{\partial a} \cdot a + \frac{\partial}{\partial a}\left(\frac{\partial e^e}{\partial a} \cdot a\right) \cdot a
\end{aligned}
$$

Lecture summary

- Fluid-mixture type models & methods for compressible multicomponent flow
  1. Linear equation of state
  2. Nonlinear equation of state

- Homogeneous relaxation models & methods methods for compressible two-phase flow
  3. Reduced models
  4. Nonlinear equation of state

- Homogeneous relaxation models & methods for compressible three-phase flow & more
  5. Reduced 4-equation based models
  6. Future perspectives

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All lecture slides are available at CSRC course website:
http://www.csrc.ac.cn/en/event/schools/2017-03-31/25.html

Thank you