Dissipative dynamics from a stochastic perspective

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ABSTRACT

The stochastic description of dissipative dynamics for both quantum and classical systems is revisited by using Itô calculus. A procedure to work out the master equation for linear systems based on this scheme is outlined. For the Caldeira–Leggett model it is shown that the master equation describing pure dephasing is Markovian-like and the solution is given. The Caldeira–Leggett master equation for Ohmic dissipation, which is valid at high temperatures, is derived.

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1. Introduction

Many important physical, chemical and biological processes take place in condensed phases and they are strongly affected by the inevitable couplings with the media. Even these processes are well understood when they are isolated from surroundings say in gas phase, their mechanism usually cannot be used to model the condensed-phase processes. In the extreme case, the gas phase process may not exist at all in condensed media or visa versa. Because the whole system consists of gigantic number of degrees of freedom and only few are ruling the process we are interested in, there is no way and also no need to describe the full dynamics. For simplicity, we call the few variables the system whereas the rest the heat bath or environment. It was Einstein who first gave a theoretical explanation of the most popular condensed-phase dynamics–Brownian motion and thus laid the foundation for classical dynamics of dissipative systems and fascinated mathematicians to discover a new field, that is, stochastic processes. Since Einstein, many scientists made several breakthroughs in the field, the generalized fluctuation-dissipation theorem, Kramers theory, stochastic resonance, the fluctuation theorem to name but a few. Nowadays there are many excellent monographs and review articles on the dynamics of dissipative systems, focusing on different respects and methodologies [1–13]. As research on quantum information attracts more and more efforts in recent years, quantum dissipation, the curse of quantum effect, becomes a very hot research topic.

In classical cases, there are well established methods such as the Langevin equation and Fokker–Planck equation techniques. These approaches have been successfully used to study the

Brownian motion and thus many intriguing phenomena including turnover effect in the over barrier process [14] and stochastic resonance [10] have been explained. By contrast, quantum dissipation is still a hard problem even today although several useful schemes have been proposed. For instance, the projection operator method was frequently used in the community of spectroscopy and quantum optics where dissipation is weak and perturbation approximation is applicable [1,12]. In this framework a generalized Langevin equation can be derived, in which the variables of the system are projected out while the environment affects the motion of the system *formally* by inducing random force and self-interaction on the system. Unfortunately, the environmental degrees of freedom still appear in operator form in the generalized Langevin equation. Another scheme is the influence functional method, which is founded by Feynman and Vernon [15] and extensively exploited by Caldeira and Leggett and many others [16,5,9]. Based on path integral, the influence functional approach has proven to be a very powerful tool for theoretical analysis of quantum dissipation. The main idea of this scheme is to derive a path-integral expression of the reduced density matrix in which environmental degrees are totally traced over, resulting in an influence functional. For harmonic-oscillator heat bath with linear couplings, Feynman and Vernon found that the influence functional is Gaussian [15]. Caldeira and Leggett [16] discussed why this environment can be used as a generic one. In the light of influence functional methodology the dynamics as well as thermodynamics of the dissipative two-state system or the spin-boson model has been extensively explored [5,9] although a full answer to the case of strong dissipation and low temperatures is still elusive. Note that the influence functional approach only allows one to write down a formal expression of the reduced density matrix in path integral. A real solution for general systems except the linear ones is not possible. The

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situation is not alleviated in numerics, thus one is only able to simulate the short-time dynamics or the system with short-ranged self-interaction induced by bath [17,18].

In Langevin equation the random force is the indispensable role that the implicit heat bath plays. For the harmonic oscillator bath Feynman and Vernon [15] recognized that one part of the influence functional is equivalent to the stochastic correlation of certain random fields, which means that the environment imposes noises on the system as it does in classical mechanics. This fact offers the basis of a stochastic formulation for quantum dissipative dynamics [19]. Recently we have proven that the heat bath with general interaction can be rigorously represented by random fields, which naturally leads to quantum stochastic Liouville equation. Unlike the classical counterpart in the Langevin equation these bath-induced fields are complex noises [20]. Therefore, the dissipative dynamics is now described by the corresponding stochastic Liouville equation whose statistical expectation gives the exact reduced density matrix. Such a stochastic theory provides not only analytic machinery but also a simulation approach. We observed that a direct numeric implementation of this scheme in general works for short times because the statistical errors grow very fast. For Ornstein–Uhlenbeck noises, however, a deterministic, hierarchical algorithm based on the stochastic scheme was proposed to solve quantum dissipative dynamics effectively [21]. Motivated by the success, we recently started from the stochastic description and took advantages of both the random and deterministic treatments to develop a mixed strategy. The obtained mixed deterministic-stochastic technique was used to solve the dynamics of the spin-boson model with strong dissipation at zero temperature, which could not be solved by any other known methods [22].

In this paper, we will first show how to use Itô calculus instead of Hubbard–Stratonovich transformation to decouple the interaction between the system and the heat bath in Section 2. This can be regarded as an alternative derivation of the stochastic scheme. The Girsanov theorem is then used to obtain the required stochastic equation of motion for the reduced density matrix or stochastic Liouville equation. In Section 3 we will review the formal solution of the stochastic Liouville equation, which was discussed in the second paper of Ref. [20], and outline how to acquire the exact master equation for linear systems. Different semiclassical approximations will be addressed in Section 4. We will summarize the results and make some conclusions in Section 5.

2. Stochastic description of dissipative dynamics

Hubbard–Stratonovich transformation has been a powerful technique to get ride of two-body interactions in the calculation of partition functions and real-time dynamics for many-body systems [23,24]. When the stochastic description of quantum dissipative dynamics was first proposed, we used the Hubbard–Stratonovich transformation in the representation of path integral to decouple interactions between the system and the heat bath during the time evolution [20]. This manipulation is not a necessary step in the scheme. In fact, decoupling can also be realized by virtue of Itô calculus and stochastic averaging. To this end we consider the whole system defined by the Hamiltonian $H = H_s + H_b + f(\hat{s})g(\hat{b})$ where H_s and H_b are the Hamiltonians of renormalized system of interest and the heat bath, respectively, and the last term indicates the interaction. Note that the whole system obeys the quantum Liouville equation,

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho], \qquad (1)$$

where $\rho(t)$ is the density matrix. For simplicity, we assume an factorized initial condition $\rho(0) = \rho_s(0)\rho_b(0)$, namely, the system and the environment are initially unentangled. Suppose that one can

establish two separate equations of motion for a random system and a random bath with initial conditions $\rho_s(0)$ and $\rho_b(0)$, respectively. Then, no matter how the equations for the random system and bath look like, the decoupling is valid when the stochastic expectation of $\rho_s(t)\rho_b(t)$ satisfies the Liouville equation (1). Now consider the following two stochastic differential equations

$$i\hbar d\rho_s = [H_s, \rho_s]dt + \frac{\sqrt{\hbar}}{2}[f(\hat{s}), \rho_s]dW_1 + i\frac{\sqrt{\hbar}}{2}\{f(\hat{s}), \rho_s\}dW_2^*, \qquad (2)$$

and

$$i\hbar d\rho_b = [H_b, \rho_b]dt + \frac{\sqrt{\hbar}}{2}[g(\hat{b}), \rho_b]dW_2 + i\frac{\sqrt{\hbar}}{2}\{g(\hat{b}), \rho_b\}dW_1^*, \qquad (3)$$

where $W_1(t) = \int_0^t [\mu_1(t') + i\mu_4(t')]dt'$ and $W_2(t) = \int_0^t [\mu_2(t') + i\mu_3(t')]dt'$ are two complex-valued Wiener processes with $\mu_j(t) (j = 1-4)$ being the uncorrelated white noises. Let M denote the statistical average. Then there are [25] $M\{\mu_j(t)\} = 0$ and $M\{\mu_j(t)\mu_k(t')\} = \delta_{jk}\delta(t - t')$. Note that in the conventional Wiener process there is only one white noise $W(t) = \int_0^t \mu_1(t')dt'$ and dW is roughly in the order of $\sqrt{dt}$. As a result, Itô calculus says that $(dW)^2 = dt$ and $dWdW' = 0$ if $W'(t) = \int_0^t \mu_2(t')dt'$ is another Wiener process. These rules can be straightforwardly extended to the complex process and there are $dW_j dW_k = dW_j^* dW_k^* = 0$ and $dW_j dW_k^* = 2\delta_{jk}dt$. With these properties one readily shows that the stochastic averaging $M\{\rho_s(t)\rho_b(t)\}$ of the product of the solutions of Eqs. (2) and (3) indeed satisfies the Liouville equation (1). That is, one has proven $\rho(t) = M\{\rho_s(t)\rho_b(t)\}$. One wonders if there is a simpler decoupling scheme by using the conventional, real Wiener processes. The answer is affirmative. For instance, the following pair of stochastic differential equations is an example,

$$i\hbar d\rho_s = [H_s, \rho_s]dt + \sqrt{\hbar/2}[f(\hat{s}), \rho_s]dW + i\sqrt{\hbar}/2\{f(\hat{s}), \rho_s\}dW',$$

and

$$i\hbar d\rho_b = [H_b, \rho_b]dt + \sqrt{\hbar/2}[g(\hat{b}), \rho_b]dW' + i\sqrt{\hbar}/2\{g(\hat{b}), \rho_b\}dW.$$

But the simplicity is only superficial because solving these equations will be more difficult. In the following, therefore, only Eqs. (2) and (3) will be used. It is clear that the system and the environment are no longer coupled with each other through "quantum" interaction and the dynamics can be explored in smaller spaces of the system and the bath separately. Does it mean that the couplings between the system and the heat bath are absolutely removed? No, it does not. In the stochastic description the interactions between the system and the environment are converted into the correlation during their evolution because both the system and bath are subject to the common random noises. When one wants to simulate the dynamics of the whole systems, in principle, one can solve these stochastic differential equations and calculate the statistical expectation of the product of two solutions to obtain the density matrix of the totally system, $\rho(t) = M\{\rho_s(t)\rho_b(t)\}$. This procedure is of course exact when these two steps can be done exactly. But our aim is to study the dynamics of the system, thus it is sufficient to calculate the reduced density matrix $\tilde{\rho}_s(t) = \text{Tr}_b\{\rho(t)\} = M\{\rho_s(t)\text{Tr}_b[\rho_b(t)]\}$ instead of $\rho(t)$ itself. We thus need to calculate the trace $\text{Tr}_b\rho_b(t)$ of the random reduced density matrix. Unlike physical systems defined by Hermitian Hamiltonians, the random heat bath is characterized by non-Hermitian Hamiltonian, and the trace of the density matrix is in general time dependent and not unity. Formally its formal expression can be written down, namely,

$$\text{Tr}_b\rho_b(t) = \exp\left[\int_0^t \bar{g}(t')dW_1^*/\sqrt{\hbar}\right],$$

where $\bar{g}(t) = \text{Tr}_b\{\rho_b(t)g(\hat{b})\}/\text{Tr}_b\rho_b(t)$. To go further a crucial step is to apply the Girsanov transformation [26], that is, replacing $W_1(t)$

by $W_1(t) + \int_0^t dt' \bar{g}(t')/\sqrt{\hbar}$. As a result, the trace of $\rho_b(t)$ is absorbed into the probability density of the white noises and the exact reduced density matrix is thus the statistical expectation of a new random $\rho_s(t)$ obeying

$$i\hbar d\rho_s = [H_s + \bar{g}(t)f(\hat{s}), \rho_s]dt + \sqrt{\hbar}/2[f(\hat{s}), \rho_s]dW_1$$
$$+ i\sqrt{\hbar}/2\{f(\hat{s}), \rho_s\}dW_2^*. \qquad (4)$$

Similar to the influence functional in the path integral treatment [16], $\bar{g}(t)$ is the bath-induced field and it exactly determines the action of the environment on the system. The equation may be called a stochastic Liouville equation and can be regarded as the quantum analog of the Langevin equation in the sense that the heat bath in both of them is implicit and its couplings to the system result in stochastic fields. One should note that Kubo [27] was the first to introduce the stochastic Liouville equation by mere considering a random interaction term in dealing with the Liouville equation. He based his discussion on physical intuition and his stochastic Liouville equation is different from the rigorously derived Eq. (4) from a microscopic model. It will become clear that it is two Hamiltonians instead of one that dictate the underlying dynamics in Eq. (4). As one can think out, when the two Hamiltonians are taken to be the same approximately Kubo's result is recovered. Calculating the statistical expectation is normally not a light task. Then, we wonder whether there is a master equation for the reduced density matrix. If there were, things would be perfect. Let us look at Eq. (4). Taking the statistical expectation on both sides gives $\tilde{\rho}_s(t)$,

$$i\hbar \frac{\partial \tilde{\rho}_s}{\partial t} = [H_s, \tilde{\rho}_s] + [f(\hat{s}), M\{\bar{g}(t)\rho_s\}]. \qquad (5)$$

It is an exact equation for the reduced density matrix, though, it is not in closed form because the relation between $\tilde{\rho}_s(t)$ and $M\{\bar{g}(t)\rho_s(t)\}$ is in general unknown. Therefore, one cannot expect a general master equation. We would like to point out that the discussion above is still applicable when the system is driven by external fields.

The decoupling procedure can be directly applied to the study of classical dissipation as well. Now the probability density in phase space $(\vec{q}, \vec{p})$ of the whole system satisfies classical Liouville equation $\partial \rho_{cl}/\partial t = \{H, \rho_{cl}\}_{PB}$ where the Poisson bracket is defined as $\{u, v\}_{PB} = \nabla_{\vec{q}} u \cdot \nabla_{\vec{p}} v - \nabla_{\vec{p}} u \cdot \nabla_{\vec{q}} v$. The classical analogues of Eqs. (2) and (3) become

$$d\rho_{s,cl} = \left\{H_s, \rho_{s,cl}\right\}_{PB} dt + \frac{1}{2}\left\{f(s), \rho_{s,cl}\right\}_{PB} dW_1 + \frac{1}{2}f(s)\rho_{s,cl}dW_2^*, \qquad (6)$$

and

$$d\rho_{b,cl} = \left\{H_b, \rho_{b,cl}\right\}_{PB} dt + \frac{1}{2}\left\{g(b), \rho_{b,cl}\right\}_{PB} dW_2 + \frac{1}{2}g(b)\rho_{b,cl}dW_1^*. \qquad (7)$$

As the reduced probability density $\tilde{\rho}_{s,cl}(t)$ is the integral of $\rho_{cl}(t)$ over the phase space of the heat bath, one first obtains the integral of the random probability density $\rho_{b,cl}$ determined by Eq. (7) and then resorts to the Girsanov transformation to obtain the classical counterpart of Eq. (4)

$$d\rho_{s,cl} = \left\{H_s + \bar{g}_{cl}(t)f(s), \rho_{s,cl}\right\}_{PB} dt + \frac{1}{2}\left\{f(s), \rho_{s,cl}\right\}_{PB} dW_1 + \frac{1}{2}f(s)\rho_{s,cl}dW_2^*, \qquad (8)$$

where the bath-induced random field reads

$$\bar{g}_{cl}(t) = \frac{\int d\vec{q}_b d\vec{p}_b \rho_{b,cl}(t)g(b)}{\int d\vec{q}_b d\vec{p}_b \rho_{b,cl}(t)}.$$

Given $\rho_{s,cl}(t)$, one takes the statistical expectation to have $\tilde{\rho}_{s,cl}(t) = M\{\rho_{s,cl}(t)\}$, which is parallel to the quantum case.

3. Caldeira–Leggett model and solution for pure dephasing

Caldeira–Leggett model is a generic microscopic model describing quantum dissipation. In this model the heat bath consists of independent harmonic oscillators and the interactions with the system are linear, $H_b = \sum_j (\hat{p}_j^2/2m_j + m_j\omega_j^2\hat{x}_j^2/2)$ and $g(\hat{b}) = \sum_j c_j\hat{x}_j$. The renormalized term of the system due to dissipation is implicitly taken in H_s in the following treatment. Suppose the environment starts from the equilibrium state. One can use either the influence functional or solve the equation of motion for the random environment to obtain the environment-induced field [20],

$$\bar{g}(t) = \sqrt{\hbar} \int_0^t \left[\alpha_R(t - t')dW_1^* + \alpha_I(t - t')dW_2\right], \qquad (9)$$

where $\alpha_R(t)$ and $\alpha_I(t)$ are the real and imaginary parts of the memory function,

$$\alpha(t) = \sum_j \frac{c_j^2}{2m_j\omega_j} \left[\coth\left(\frac{\hbar\beta\omega_j}{2}\right)\cos(\omega_j t) - i\sin(\omega_j t)\right], \qquad (10)$$

which is the autocorrelation function of the "force" $g(\hat{b})$. For the environment consisting of a continuum of harmonic oscillators the spectral density function

$$J(\omega) = \frac{\pi}{2} \sum_j \frac{c_j^2}{m_j\omega_j}\delta(\omega - \omega_j)$$

fully defines the bath-induced stochastic field, namely,

$$\alpha(t) = \frac{1}{\pi} \int_0^\infty d\omega J(\omega)\left[\coth\left(\frac{\hbar\beta\omega}{2}\right)\cos(\omega t) - i\sin(\omega t)\right]. \qquad (11)$$

Upon inserting into Eq. (5), the equation of motion for the reduced density matrix reads

$$i\hbar \frac{\partial \tilde{\rho}_s}{\partial t} = [H_s, \tilde{\rho}_s] + [f(\hat{s}), \hat{O}(t)], \qquad (12)$$

where the new operator $\hat{O}(t)$ results from the correlation of the noise and the random density matrix itself, modified by the response function $\alpha(t)$,

$$\hat{O}(t) = \sqrt{\hbar} \int_0^t dt' \Big[\alpha_R(t - t')M\{\rho_{s,1}(t, t')\}$$
$$+ \alpha_I(t - t')M\{\rho_{s,2}(t, t')\}\Big]. \qquad (13)$$

One immediately reads off the random quantities in the expression, $\rho_{s,1}(t, t') = [\mu_1(t') - i\mu_4(t')]\rho_s(t)$ and $\rho_{s,2}(t, t') = [\mu_2(t') + i\mu_3(t')]\rho_s(t)$. Here in the derivation the nonanticipating property of $\rho_s(t)$,

$$M\{\rho_s(t)dW_{1(2),t}\} = 0,$$

namely, that $\rho_s(t)$ is independent of the increment $dW_{1,2}$ of $W_{1,2}$ at time t is used. Now we divide the operator $\hat{O}(t)$ into two parts corresponding to the contribution of the real and imaginary parts of the response function $\alpha(t)$, respectively, $\hat{O}(t) = \hat{O}_R(t) + \hat{O}_I(t)$. Then, applying the Novikov theorem [28], that is, $M\{\mu(t')F[\mu]\} = M\{\delta F/\delta\mu(t')\}$ for a white noise μ and its functional $F[\mu]$, one obtains

$$M\{\rho_{s,1}(t, t')\} = M\left\{\frac{\delta\rho_s(t)}{\delta\mu_1(t')} - i\frac{\delta\rho_s(t)}{\delta\mu_4(t')}\right\} \equiv M\{\hat{O}_{s,1}(t, t')\}, \qquad (14)$$

$$M\{\rho_{s,2}(t, t')\} = M\left\{\frac{\delta\rho_s(t)}{\delta\mu_2(t')} + i\frac{\delta\rho_s(t)}{\delta\mu_3(t')}\right\} \equiv M\{\hat{O}_{s,2}(t, t')\}. \qquad (15)$$

Substituting into Eq. (12) yields

$$i\hbar \frac{\partial \tilde{\rho}_s}{\partial t} = [H_s, \tilde{\rho}_s] + [f(\hat{s}), M\{\hat{O}_R(t)\} + M\{\hat{O}_I(t)\}], \qquad (16)$$

where

$$\widehat{O}_{R(I)}(t) = \sqrt{\hbar} \int_0^t dt' \alpha_{R(I)}(t-t') \widehat{O}_{s,1(2)}(t,t').$$

Similar to the ordinary Liouville equation, the stochastic Liouville equation Eq. (4) is linear and assumes a formal solution, $\rho_s(t) = U_1(t,t_0)\rho_s(t_0)U_2(t_0,t)$ where the two time-evolution propagators $U_{1,2}(t,t_0)$ correspond to two *different* stochastic Hamiltonians

$$H_1(t) = H_s + \left\{ \bar{g}(t) + \frac{\sqrt{\hbar}}{2}[\mu_1(t) + i\mu_2(t) + \mu_3(t) + i\mu_4(t)] \right\} f(\hat{s}), \quad (17)$$

$$H_2(t) = H_s + \left\{ \bar{g}(t) + \frac{\sqrt{\hbar}}{2}[\mu_1(t) - i\mu_2(t) - \mu_3(t) + i\mu_4(t)] \right\} f(\hat{s}). \quad (18)$$

This is the essential difference between Eq. (4) and the stochastic Liouville equation proposed phenomenologically by Kubo, which is ruled by "one" random Hamiltonian. One may work in Hilbert space instead of the larger Liouville space either for convenience of numerical simulation or a physical interpretation of quantum measurement [29]. To this end the density matrix should be unraveled. This can be readily done for a pure initial state, i.e., $\rho_s(0) = |\psi_s(0)\rangle\langle\psi_s(0)|$. Using the two propagators, one naturally obtains $\tilde{\rho}_s(t) = M\{|\psi_{s,1}(t)\rangle \langle\psi_{s,2}(t)|\}$, where $|\psi_{s,1(2)}(t)\rangle = U_{1(2)}(t,0)|\psi_s(0)\rangle$.

By virtue of the formal solution of the two propagators, the functional derivatives in Eqs. (14) and (15) can be expressed, respectively, as

$$\widehat{O}_{s,1}(t,t') = -\frac{i}{\sqrt{\hbar}}[U_1(t,t')f(\hat{s})U_1(t',t)\rho_s(t)$$
$$- \rho_s(t)U_2(t,t')f(\hat{s})U_2(t',t)] \quad (19)$$

and

$$\widehat{O}_{s,2}(t,t') = \frac{1}{\sqrt{\hbar}}[U_1(t,t')f(\hat{s})U_1(t',t)\rho_s(t)$$
$$+ \rho_s(t)U_2(t,t')f(\hat{s})U_2(t',t)]. \quad (20)$$

One can also show that the other pair of functional derivatives is related to these expression via

$$\frac{\delta\rho_s(t)}{\delta\mu_1(t')} + i\frac{\delta\rho_s(t)}{\delta\mu_4(t')} = 2\int_{t'}^t dt_1 \alpha_R(t_1-t')\widehat{O}_{s,1}(t,t_1) \quad (21)$$

and

$$\frac{\delta\rho_s(t)}{\delta\mu_2(t')} - i\frac{\delta\rho_s(t)}{\delta\mu_3(t')} = 2\int_{t'}^t dt_1 \alpha_I(t_1-t')\widehat{O}_{s,2}(t,t_1). \quad (22)$$

When dissipation causes pure dephasing [30], there is no energy exchange between the system and the bath, which means $[H_s, f(\hat{s})] = 0$. In this case there are $[U_{1,2}(t,t'), f(\hat{s})] = 0$, which directly lead to $\widehat{O}_{s,1}(t,t') = -i/\sqrt{\hbar}[f(\hat{s}), \rho_s(t)]$ and $\widehat{O}_{s,2}(t,t') = 1/\sqrt{\hbar}\{f(\hat{s}), \rho_s(t)\}$. Inserting into Eq. (16), we obtain the master equation for pure dephasing,

$$i\hbar\frac{\partial\tilde{\rho}_s}{\partial t} = [H_s, \tilde{\rho}_s] - iC_R(t)[f(\hat{s}), [f(\hat{s}), \tilde{\rho}_s]] + C_I(t)[f^2(\hat{s}), \tilde{\rho}_s], \quad (23)$$

where $C_{R,I}(t) = \int_0^t dt' \alpha_{R,I}(t')$. This equation is of the same form as that from Markovian approximation for arbitrary $f(\hat{s})$, which also reflects the short-time universality of decoherence [31]. Now let $|m\rangle$ be the common eigenvectors of H_s and $f(\hat{s})$, that is, $H_s|n\rangle = E_n|n\rangle$ and $f(\hat{s})|n\rangle = f_n|n\rangle$. Represented in term of eigenvectors, Eq. (23) becomes very simpler,

$$i\hbar\frac{\partial\tilde{\rho}_{s,n'n}}{\partial t} = \left[E_{n'} - E_n - iC_R(t)(f_{n'} - f_n)^2 + C_I(t)(f_{n'}^2 - f_n^2)\right]\tilde{\rho}_{s,n'n}, \quad (24)$$

where $\tilde{\rho}_{s,n'n} = \langle n'|\tilde{\rho}_s|n\rangle$. The solution reads

$$\tilde{\rho}_{s,n'n}(t) = \tilde{\rho}_{s,n'n}(0) \exp\left\{\frac{i}{\hbar}\left[(E_n - E_{n'})t + \Gamma_I(t)(f_n^2 - f_{n'}^2)\right.\right.$$
$$\left.\left. + i\Gamma_R(t)(f_{n'} - f_n)^2\right]\right\},$$

where $\Gamma_I(t) = \int_0^t dt' C_I(t')$ and $\Gamma_R(t) = \int_0^t dt' C_R(t')$. One immediately observes that the diagonal elements $\tilde{\rho}_{s,nn}$ do not change with time and the non-diagonal terms do. It is also clear that only $C_R(t)$ from the real part of response function of the heat bath effects the decay of $\tilde{\rho}_{s,n'n}(t)$.

Because the dissipative harmonic oscillator acts as an important model system for understanding specific features of quantum dissipation, the master equation has been derived over and over again by using different methods [32]. The same master equation can also be obtained by virtue of the stochastic description. To this end one first solves the stochastic propagators and then calculates the statistical expectations of the functional derivatives given by Eqs. (19)–(22). For linear systems the involved operators in these equations assume fixed forms and their coefficient functions are determined by a set of integral equations. We will detail the derivation of the master equation for a driven dissipative harmonic oscillator elsewhere [33].

We would like to point out that starting from the Liouville equation of the whole system, one can develop different schemes and perspectives for understanding quantum dissipation. An interesting treatment was proposed by Cao open systems. one can develop diversified schemes for treating quantum dissipation

4. High-temperature approximations

Given the initial state of the system and the bath-induced stochastic field $\bar{g}(t)$, one can solve Eq. (4) and then obtain the reduced density matrix with sufficient number of random realizations. Because the system of interest is in general a small one, it is not hard to solve its exact quantum dynamics. Note, however, that in our case the system is subject to the stochastic field the environment effects and one has to calculate many "quantum" samplings to reach the convergent statistical expectation of the reduced density matrix. Roughly speaking, it is the statistics instead of quantum dynamics that makes the problem difficult. At high temperatures or in classical regime, the bath-induced stochastic field is more local in a sense that the self-interaction of the system due to the bath is short-ranged, one may resort to the Markovian approximation. We again consider the Caldeira–Leggett model, see Eq. (9). Note that in the bath-induced field the imaginary part of the memory function is independent of temperature, which is of a genuine quantum effect. At high temperature (small β) we use Taylor expansion for $\coth(\hbar\beta\omega)$ in the real part of the memory function to obtain the dominant term

$$\alpha_R(t) = \frac{2}{\pi}\int_0^\infty d\omega \frac{J(\omega)}{\hbar\beta\omega}\cos(\omega t).$$

In Ohmic dissipation $J(\omega) = \eta\omega$ one finds that $\alpha_{R,I}(t)$ behave as distribution functions

$$\alpha_R(t) = \frac{2\eta}{\hbar\beta}\delta(t)$$

and

$$\alpha_I(t) = \eta\delta'(t).$$

It is a bit subtle to use these distribution functions because they come out in the integral from 0 to t. In other words, only half of the distribution contributes to the integration. Without a rigorous proof we simply put down as

$$\int_0^t dt' \delta(t') f(t') = \frac{1}{2} f(0),$$

$$\int_0^t dt' \delta'(t') f(t') = -\frac{1}{2} f'(0).$$

These expressions can be substituted into Eq. (16) to obtain

$$\widehat{O}_R(t) = \frac{\eta}{\sqrt{\hbar \beta}} \widehat{O}_{s,1}(t,t), \tag{25}$$

$$\widehat{O}_I(t) = \frac{\eta}{2} \sqrt{\hbar} \frac{\partial}{\partial t'} \widehat{O}_{s,2}(t,t')|_{t'=t}, \tag{26}$$

with the help of Eqs. (19) and (20) we readily acquire

$$M\left\{\widehat{O}_R(t)\right\} = -\frac{i\eta}{\hbar \beta}[f(\hat{s}), \tilde{\rho}_s(t)], \tag{27}$$

$$M\left\{\widehat{O}_I(t)\right\} = \frac{i\eta}{2\hbar}\{[H_s, f(\hat{s})], \tilde{\rho}_s(t)\}. \tag{28}$$

Inserting into Eq. (16), we obtain the semiclassical master equation

$$ih\frac{\partial \tilde{\rho}_s}{\partial t} = [H_s, \tilde{\rho}_s] - \frac{i\eta}{\hbar \beta}[f(\hat{s}), [f(\hat{s}), \tilde{\rho}_s]] + \frac{i\eta}{2h}[f(\hat{s}), \{[H_s, f(\hat{s})], \tilde{\rho}_s\}]. \tag{29}$$

For conventional systems $H_s = \hat{p}^2/2M + V_R(\hat{x})$ where the potential operator is the renormalized one and $f(\hat{s}) = \hat{x}$ there is $[H_s, f(\hat{s})] = -ih\hat{p}/M$. Inserting into Eq. (29) leads to the well-known Caldeira–Leggett master equation

$$ih\frac{\partial \tilde{\rho}_s}{\partial t} = [H_s, \tilde{\rho}_s] - \frac{i\eta}{\hbar \beta}[\hat{x}, [\hat{x}, \tilde{\rho}_s]] + \frac{\eta}{2M}[\hat{x}, \{\hat{p}, \tilde{\rho}_s\}]. \tag{30}$$

This equation has been frequently discussed from different perspectives in the literature, for instance, in Ref. [34]. Note that in this approximation the system follows the exact quantum dynamics while the environment is rather "classical" as it should be at high temperatures. In this case the dissipative dynamics is Markovian. To study low-temperature behavior it is required to develop an efficient method to reveal the non-Markovian effect [35].

5. Summary

The paper provides a different way to understand the stochastic formulation of dissipative systems. It conveys the idea that decoupling the interactions between the subsystems is possible by virtue of Itô calculus, which leads to a stochastic formulation of the involved dynamics and might be a convenient way to study complex systems. When one investigates the dynamics of the dissipative system, making such a change from the deterministic to stochastic perspective is especially helpful. By doing so the environment is totally reflected in the noise it induces in the evolution of the system. Then the exact dynamics is the statistical expectation of the random motion of the system. This observation holds for both quantum and classical systems. We reviewed the formal solution for the Caldeira–Leggett model and presented the solution for pure dephasing. Besides, the procedure to acquire the exact master equation for linear systems was briefly outlined and the semiclassical Caldeira–Leggett master equation was derived. It should be stressed that nonlocal or memory effects due to the heat bath which are important at low temperatures are lost in this approximation. To obtain more accurate dynamics it is necessary to find a good means of dealing with the memory effect.

Acknowledgments

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