Non-Markovian Relaxation of a Three-Level System: Quantum Trajectory Approach

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The non-Markovian dynamics of a three-level quantum system coupled to a bosonic environment is a difficult problem due to the lack of an exact dynamic equation such as a master equation. We present for the first time an exact quantum trajectory approach to a dissipative three-level model. We have established a convolutionless stochastic Schrödinger equation called the time-local quantum state diffusion (QSD) equation without any approximations, in particular, without Markov approximation. Our exact time-local QSD equation opens a new avenue for exploring quantum dynamics for a higher dimensional quantum system coupled to a non-Markovian environment.

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Every small quantum system, such as a two-level atom (qubit), a three-level atom (qutrit), or a cavity mode (simple harmonic oscillator), should be regarded as an open quantum system due to the inevitable interaction with its environment. The effect from the environment may bring about some grave problems, such as quantum decoherence and disentanglement, to the dynamics of the open system [1]. Much has been studied in the Markov limit where the dynamics of open quantum system is typically described by a standard Lindblad master equation which is equivalent to a quantum state diffusion (QSD) equation for a pure state [2–7] (setting $\hbar = 1$):

$$\frac{d}{dt}\psi_t = -iH\psi_t + \Delta_t(L)(z_t + \langle L^\dagger\rangle_t)\psi_t - \frac{1}{2}\Delta_t(L^\dagger L)\psi_t, \tag{1}$$

where the notation $\Delta_t(A) \equiv A - \langle A\rangle_t$ for any operator A and $\langle A\rangle_t \equiv \langle\psi_t|A|\psi_t\rangle$ denotes the quantum average. L is the system operator coupled to the environment, often called the Lindblad operator. The dynamics of the open system is driven by the classical stochastic process z_t. The reduced density matrix for the system of interest can be recovered by averaging quantum trajectories generated by the QSD equation (1): $\rho_t = M[|\psi_t(z)\rangle\langle\psi_t(z)|]$. Here $M[\cdot]$ denotes the ensemble average over the classical noise. Besides many appealing features exhibited by pure state quantum trajectories, Eq. (1) provides a very efficient numerical tool in solving quantum dynamics of Markov open systems.

Non-Markovian environments have become increasingly important in recent times due to their relevance in explaining new experimental advances in high-Q microwave cavities, photonic crystals, and atom laser in Bose-Einstein condensation [8–14]. It is also evident from the recent progress in quantum information processing that environmental memory may be utilized to generate or modulate entanglement evolution of an open quantum system [15,16]. Clearly, an approach that is capable of dealing with non-Markovian quantum system is highly desirable. A non-Markovian QSD equation for a general non-Markovian open system by Diósi and co-workers has provided a powerful tool in dealing with the exact dynamics of quantum open systems irrespective of the coupling strength, the correlation time, and the spectral density of the bosonic environments [17–24]. Despite extensive research, the explicit non-Markovian QSD equations only exist for a single two-level system, the quantum Brownian motion model, and optical cavities due to the intricate functional derivative appearing in the fundamental QSD equation [17,18,25–28]. Clearly, the power of the non-Markovian quantum trajectory method cannot be readily unleashed unless the stochastic QSD equation can be cast into a convolutionless form in which the numerical simulations and analytical applications can be easily implemented [29,30].

In this Letter, we present, for the very first time, an exact time-local QSD equation for the three-level dissipative dynamics in the framework of non-Markovian quantum trajectory at zero temperature. Our treatments of the three-level model have opened up a new route of exploring dynamics of higher dimensional quantum open systems. Notably, our method of establishing a time-local non-Markovian QSD equation can be applied to multiple qubit systems to deal with the time behavior of them such as the estimation of non-Markovian entanglement evolution and fidelity [31,32].

We consider a Caldeira-Leggett-like model [28] involving a system with Hamiltonian H describing a three-level atom, coupled linearly to a general boson environment consisting of a set of harmonic oscillators $a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger$ (e.g., cavity modes). The total Hamiltonian for the system of interest plus environment can be written as

$$H_{\rm tot} = H + \sum_{\mathbf{k}}(g_{\mathbf{k}}^* L^\dagger a_{\mathbf{k}} + g_{\mathbf{k}} L a_{\mathbf{k}}^\dagger) + \sum_{\mathbf{k}}\omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}, \tag{2}$$

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where the three-level system Hamiltonian $H = \omega J_z = \omega(|2\rangle\langle 2| - |0\rangle\langle 0|)$, the Lindblad operator $L = J_- = \sqrt{2}(|0\rangle\langle 1| + |1\rangle\langle 2|)$, and $L^\dagger = J_+$. Note that ω is the spacing of the two neighboring energy levels of the three-level system, $g_{\mathbf{k}}$ are the coupling constants between the three-level system and the environmental modes. It is known [17] that the linear QSD equation for the three-level system can be formally written as

$$\frac{d}{dt}\psi_t(z) = -i\omega J_z\psi_t(z) + J_- z_t\psi_t(z) - J_+\int_0^t ds\alpha(t,s)\frac{\delta\psi_t(z)}{\delta z_s}, \quad (3)$$

where $\alpha(t,s)$ is the bath correlation function and $z_t = -i\sum_{\mathbf{k}} g_{\mathbf{k}}^* z_{\mathbf{k}}^* e^{i\omega_{\mathbf{k}}t}$ is a complex Gaussian process satisfying $M[z_t] = M[z_t z_s] = 0$ and $M[z_t^* z_s] = \alpha(t,s)$. When $\alpha(t,s) = \delta(t-s)$, it reduces to the memoryless Markov case. The density matrix of the system is recovered from the ensemble average of many realizations of quantum trajectories:

$$\rho_t = M[|\psi_t(z)\rangle\langle\psi_t(z)|] = \int\frac{d^2z}{\pi}e^{-|z|^2}|\psi_t(z)\rangle\langle\psi_t(z)|. \quad (4)$$

One advantage of using quantum trajectory is that, while deriving the non-Markovian master equation is known to be a daunting task, the QSD equation governing the non-Markovian open system can be read off directly once the Hamiltonian of the total system is given by (2). However, the formal non-Markovian QSD equation (3) or its nonlinear version (see below) cannot be easily implemented as an analytical or numerical tool unless the term containing the functional derivative can be written as $\frac{\delta}{\delta z_s}\psi_t(z) = O(t,s,z)\psi_t(z)$, where $O(t,s,z)$ is an operator acting on the system Hilbert space satisfying the initial condition $O(s,s,z) = J_-$. The equation of motion governing the O operator (consistency condition [18]) is given by

$$\partial_t O(t,s,z) = [-i\omega J_z + J_- z_t - J_+\bar{O}(t,z), O(t,s,z)] - J_+\frac{\delta\bar{O}(t,z)}{\delta z_s}, \quad (5)$$

where $\bar{O}(t,z) = \int_0^t ds\alpha(t,s)O(t,s,z)$. For the model given by (2), it can be shown that this nonlinear operator equation can be solved by the following ansatz solution:

$$O = f(t,s)J_- + g(t,s)J_z J_- + i\int_0^t p(t,s,s')z_{s'}ds'J_-^2, \quad (6)$$

where the coefficient functions $f(t,s)$, $g(t,s)$, and $p(t,s,s')$ satisfy

$$\frac{\partial}{\partial t}f(t,s) = i\omega f(t,s) + 2G(t)f(t,s) - 2iP(t,s), \quad (7)$$

$$\frac{\partial}{\partial t}g(t,s) = i\omega g(t,s) - 2F(t)f(t,s) + 2F(t)g(t,s) + 4G(t)f(t,s) - 2G(t)g(t,s) - 2iP(t,s), \quad (8)$$

$$\frac{\partial}{\partial t}p(t,s,s') = 2i\omega p(t,s,s') + 2F(t)p(t,s,s') + 2f(t,s)P(t,s') - 2g(t,s)P(t,s'), \quad (9)$$

where the time-dependent functions $F(t) \equiv \int_0^t \alpha(t,s)f(t,s)ds$, $G(t) \equiv \int_0^t \alpha(t,s)g(t,s)ds$, and $P(t,s') \equiv \int_0^t \alpha(t,s)p(t,s,s')ds$ together with a set of boundary conditions for f, g, and p: $f(s,s) = 1$, $p(t,s,t) = -ig(t,s)$, $g(s,s) = 0$, $p(s,s,s') = 0$.

With the explicit O operator (6), the exact linear non-Markovian QSD equation may be compactly written into a time-local form:

$$\frac{d}{dt}\psi_t(z) = [-i\omega J_z + J_- z_t - F(t)J_+J_- - G(t)J_+J_zJ_- - i\int_0^t P(t,s')z_{s'}ds'J_+J_-^2]\psi_t(z). \quad (10)$$

Note that the time-dependent coefficients $F(t)$, $G(t)$, and $P(t,s')$ could be calculated once the correlation function $\alpha(t,s)$ is explicitly given. It should be emphasized that the non-Markovian properties are encoded in a finite width correlation function $\alpha(t,s)$, and hence in the time-dependent coefficients $F(t)$, $G(t)$, and $P(t,s')$ appearing in the QSD equation (10). Clearly, the terms containing $G(t)$ and $P(t,s')$ give rise to the most important correction to the Markovian dynamics. This can be easily seen if we take the Markov limit in which $F(t) = 1/2$, $G(t) = P(t,s') = 0$ recovering the standard Markovian QSD equation. Equation (10) is the fundamental equation for the three-level system coupled to a bosonic environment described by (2).

For numerical simulations, it is more efficient to use the nonlinear non-Markovian QSD equation [18] for the normalized state $\tilde{\psi}_t = \frac{\psi_t}{\|\psi_t\|}$ obtained from Eq. (10):

$$\begin{aligned}\frac{d}{dt}\tilde{\psi}_t = {} & -i\omega J_z\tilde{\psi}_t + \Delta_t(J_-)\tilde{z}_t\tilde{\psi}_t + \langle J_+\rangle_t F(t)\Delta_t(J_-)\tilde{\psi}_t \\ & - F(t)\Delta_t(J_+J_-)\tilde{\psi}_t + \langle J_+\rangle_t G(t)\Delta_t(J_zJ_-)\tilde{\psi}_t \\ & - G(t)\Delta_t(J_+J_zJ_-)\tilde{\psi}_t + i\langle J_+\rangle_t \\ & \times\int_0^t P(t,s')\tilde{z}_{s'}ds'\Delta_t(J_-^2)\tilde{\psi}_t \\ & - i\int_0^t P(t,s')\tilde{z}_{s'}ds'\Delta_t(J_+J_-^2)\tilde{\psi}_t. \end{aligned} \quad (11)$$

Here $\tilde{z}_t = z_t + \int_0^t \alpha^*(t,s)\langle J_+\rangle_s ds$ is the shifted complex Gaussian process.

The correlation function $\alpha(t,s)$ at zero temperature is given by $\alpha(t,s) = \sum_{\mathbf{k}}|g_{\mathbf{k}}|^2 e^{-i\omega_{\mathbf{k}}(t-s)} = \int_0^\infty d\omega S(\omega)e^{-i\omega(t-s)}$, where $S(\omega)$ is the spectral density of the environment modes. Equation (11) is capable of

describing non-Markovian dynamics with an arbitrary correlation function $\alpha(t, s)$, but for the sake of simplicity, we now consider the correlation function $\alpha(t, s) = \frac{\gamma}{2} e^{-\gamma|t-s|}$ for the Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process is a useful approach to modeling noisy relaxation with a finite environmental memory time scale $1/\gamma$. When $\gamma \to \infty$, the environment memory time approaches zero and $\alpha(t, s)$ reduces to $\delta(t - s)$, which corresponds to the Markov limit. Then Eq. (11) reduces to the standard nonlinear QSD equation [4]:

$$\frac{d}{dt}\tilde{\psi}_t = -i\omega J_z \tilde{\psi}_t + \Delta_t(J_-)(z_t + \langle J_+\rangle_t)\tilde{\psi}_t - \frac{1}{2}\Delta_t(J_+J_-)\tilde{\psi}_t. \quad (12)$$

For the Ornstein-Uhlenbeck correlation, the partial differential equations (7)–(9) could be converted into a set of ordinary differential equations:

$$\frac{dF(t)}{dt} = \frac{\gamma}{2} + (-\gamma + i\omega)F(t) + 2F(t)G(t) - 2i\bar{P}(t), \quad (13)$$

$$\frac{dG(t)}{dt} = (-\gamma + i\omega)G(t) - 2F^2(t) + 6F(t)G(t) - 2G^2(t) - 2i\bar{P}(t), \quad (14)$$

$$\frac{d\bar{P}(t)}{dt} = -i\frac{\gamma}{2}G(t) + 2(-\gamma + i\omega)\bar{P}(t) + 4F(t)\bar{P}(t) - 2G(t)\bar{P}(t), \quad (15)$$

where $\bar{P}(t) = \int_0^t \alpha(t, s')P(t, s')ds'$ and the initial conditions are given by $F(0) = G(0) = \bar{P}(0) = 0$. It is easy to show that $P(t, s') = -iG(s')e^{\int_{s'}^{t}[-\gamma+2i\omega+4F(s)-2G(s)]ds}$.

With the nonlinear non-Markovian QSD equation (11), the simulations of the three-level system (a spin-1 particle or a three-level atom) dynamics can be efficiently implemented by realizing Gaussian sample paths. We first calculated the ensemble average of the angular momentum time evolution with different memory parameter γ. The plots are shown in Fig. 1. The zero-temperature environment prohibits the transitions from a lower level to a higher level, so for the three-level atom with an arbitrary initial state, the spontaneous emission always causes the system to decay into its ground level $|0\rangle$ in the long time limit. Therefore, when $t \to \infty$, we get $\langle J_x\rangle = \langle J_y\rangle \to 0$ and $\langle J_z\rangle \to -1$; this scenario can be easily seen from the Markov limit shown in Fig. 1(d). The non-Markovian features of the environment for different γ are illustrated in Figs. 1(a)–1(c), where we can see that the transition of dynamics from non-Markovian to Markov regimes is dictated by environment memory time $\tau = 1/\gamma$. Clearly, the non-Markovian features are lost when the system approaches its Markov limit. The most important non-Markovian corrections are dominated by $G(t)$ and the noise terms contained in the O operator of Eq. (6); they become more significant in the case of smaller γ (or longer τ). As a consequence of long environmental memory, the dynamics of $\langle J_z\rangle$ has a long "tail," which means that it needs more time to reach its final steady state compared with the case of the Markov limit [Fig. 1(d)]. The Markov dynamics emerges when τ becomes shorter and shorter; that is, for the large $\gamma \gg 1$, $G(t)$ and the noise terms can be effectively neglected.

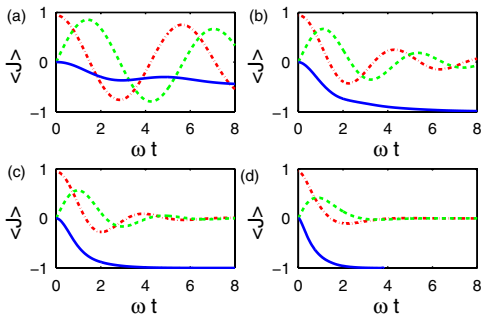

FIG. 1 (color online). Ensemble average of $\langle\vec{J}\rangle$ over 1000 realizations (red dot-dashed line for $\langle J_x\rangle$, green dashed line for $\langle J_y\rangle$, and blue solid line for $\langle J_z\rangle$) with different γ's. Here we choose $\omega = 1$ and the initial state $|\psi_0\rangle = (1/\sqrt{3})(|0\rangle + |1\rangle + |2\rangle)$. (a) $\gamma = 0.2$, (b) $\gamma = 1.0$, (c) $\gamma = 2.0$, (d) Markov.

As a measure of degree of decoherence, we now consider the purity dynamics of the three-level system as shown in Fig. 2. It can be shown that the purity varies from 1 for a pure state to $1/d$ (d is the dimension of the density matrix, here we have $d = 3$) for a maximally mixed state. We begin with a pure initial state $|\psi_0\rangle = (1/\sqrt{3})(|0\rangle + |1\rangle + |2\rangle)$, so for the zero-temperature case, the final state of the three-level system is also pure. As shown in Fig. 2, the decoherence pathways of the three-level system are profoundly modified by the environmental memory parameter γ. An interesting feature of the quantum trajectory is that it can reveal how a quantum state evolves into decoherence with only a small number of

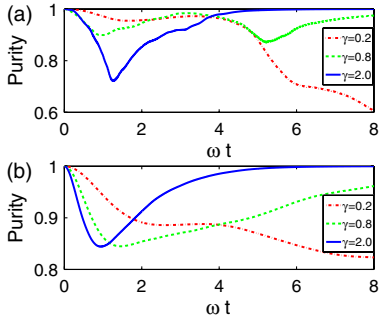

FIG. 2 (color online). Ensemble average of purity $= \mathrm{Tr}[\rho^2(t)]$ over different number of realizations, with red dot-dashed line for $\gamma = 0.2$, green dashed line for $\gamma = 0.8$, and blue solid line for $\gamma = 2.0$. Here we choose $\omega = 1$ and the initial state $|\psi_0\rangle = (1/\sqrt{3})(|0\rangle + |1\rangle + |2\rangle)$. (a) 5 trajectories and (b) 1000 trajectories.

realizations. For example, Fig. 2(a) shows the results generated by only five realizations. Clearly, one cannot expect that the five-realization simulations can quantitatively reproduce an accurate description about decoherence dynamics. Nevertheless, they still give rise to a rather interesting qualitative picture about the general dissipative behaviors of the purity including the correct information about the decay and recovery tendencies. For the numerical simulations with 1000 realizations shown in Fig. 2(b), it can be shown that a reliable physical picture can be obtained. Specifically, when $\gamma = 0.2$, the system exhibits a stronger non-Markovian feature, as such, the exchanges of quantum information (the energy distribution among the three levels $|0\rangle$, $|1\rangle$, $|2\rangle$ and the coherence between any two levels) with the environment via dissipation proceed slowly and information dissipated into the environment may come back to the system in a finite time. Consequently, the decoherence time is effectively prolonged. On the contrary, for short memory time with $\gamma = 2$, as shown in Fig. 2(b), the system quickly evolves into a mixed state and then relaxes itself to the final pure state. When $\gamma = 0.8$, the purity dynamics shows a moderate non-Markovian behavior.

In conclusion, we present an exact non-Markovian quantum state diffusion equation for the dissipative three-level model described by Eq. (2). We instigated significant and important progress by obtaining an explicit form of the O operator at zero temperature. With the time-local non-Markovian QSD equation, we are able to attack the transient property of quantum decoherence dynamics of the three-level system in all possible non-Markovian regimes and the well-known Markov limit. There are many important connections of our current work with multiple qubit and qutrit systems. A more detailed research into this subject will be useful.

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[1] H. P. Breuer and F. Petruccione, *Theory of Open Quantum Systems* (Oxford, New York, 2002).
[2] H. Carmichael, *An Open System Approach to Quantum Optics* (Springer, Berlin, 1994).
[3] J. Dalibard, Y. Castin, and K. Mölmer, Phys. Rev. Lett. **68**, 580 (1992).
[4] N. Gisin and I. C. Percival, J. Phys. A **25**, 5677 (1992); J. Phys. A **26**, 2233 (1993).
[5] C. W. Gardiner, A. S. Parkins, and P. Zoller, Phys. Rev. A **46**, 4363 (1992).
[6] H. M. Wiseman and G. J. Milburn, Phys. Rev. A **47**, 1652 (1993).
[7] For a very comprehensive review, see M. B. Plenio and P. L. Knight, Rev. Mod. Phys. **70**, 101 (1998).
[8] J. J. Hope, Phys. Rev. A **55**, R2531 (1997).
[9] S. John and T. Quang, Phys. Rev. Lett. **74**, 3419 (1995).
[10] S. Bay, P. Lambropoulos, and K. Mölmer, Phys. Rev. A **57**, 3065 (1998).
[11] N. Vats and S. John, Phys. Rev. A **58**, 4168 (1998).
[12] G. M. Moy, J. J. Hope, and C. M. Savage, Phys. Rev. A **59**, 667 (1999).
[13] I. de Vega and D. Alonso, Phys. Rev. A **77**, 043836 (2008).
[14] Y. N. Chen, G. Y. Chen, Y. Y. Liao, N. Lambert, and F. Nori, Phys. Rev. B **79**, 245312 (2009).
[15] B. Bellomo, R. Lo Franco, and G. Compagno, Phys. Rev. Lett. **99**, 160502 (2007).
[16] J. Tan, T. H. Kyaw, and Y. Yeo, Phys. Rev. A **81**, 062119 (2010).
[17] L. Diósi and W. T. Strunz, Phys. Lett. A **235**, 569 (1997).
[18] L. Diósi, N. Gisin, and W. T. Strunz, Phys. Rev. A **58**, 1699 (1998).
[19] W. T. Strunz, L. Diósi, and N. Gisin, Phys. Rev. Lett. **82**, 1801 (1999).
[20] T. Yu, L. Diósi, N. Gisin, and W. T. Strunz, Phys. Rev. A **60**, 91 (1999).
[21] W. T. Strunz, L. Diósi, N. Gisin, and T. Yu, Phys. Rev. Lett. **83**, 4909 (1999).
[22] W. T. Strunz, Chem. Phys. **268**, 237 (2001).
[23] J. Gambetta and H. M. Wiseman, Phys. Rev. A **66**, 012108 (2002); Phys. Rev. A **66**, 052105 (2002); Phys. Rev. A **68**, 062104 (2003).
[24] An interesting, but different, non-Markovian quantum trajectory approach has been developed. See, e.g., H. P. Breuer, B. Kappler, and F. Petruccione, Phys. Rev. A **59**, 1633 (1999).
[25] R. P. Feynman and F. L. Vernon, Ann. Phys. (N.Y.) **24**, 118 (1963).
[26] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).
[27] H. N. Xiong, W. M. Zhang, X. G. Wang, and M. H. Wu, Phys. Rev. A **82**, 012105 (2010).
[28] A. O. Caldeira and A. J. Leggett, Physica (Amsterdam) **121A**, 587 (1983).
[29] W. T. Strunz and T. Yu, Phys. Rev. A **69**, 052115 (2004).
[30] T. Yu, Phys. Rev. A **69**, 062107 (2004).
[31] J. Jing and T. Yu (to be published).
[32] For a Markov case, see, e.g., G. G. Carlo, G. Benenti, and G. Casati, Phys. Rev. Lett. **91**, 257903 (2003).