Chapter 25
Non-Markovian Dynamics of Qubit Systems: Quantum-State Diffusion Equations Versus Master Equations

Yusui Chen and Ting Yu

Abstract In this review we discuss recent progress in the theory of open quantum systems based on non-Markovian quantum state diffusion and master equations. In particular, we show that an exact master equation for an open quantum system consisting of a few qubits can be explicitly constructed by using the corresponding non-Markovian quantum state diffusion equation. The exact master equation arises naturally from the quantum decoherence dynamics of qubit systems collectively interacting with a colored noise. We illustrate our general theoretical formalism by the explicit construction of a three-qubit system coupled to a non-Markovian bosonic environment. This exact qubit master equation accurately characterizes the time evolution of the qubit system in various parameter domains, and paves the way for investigation of the memory effect of an open quantum system in a non-Markovian regime without any approximation.

25.1 Introduction

Recent advances in open quantum systems, quantum dissipative dynamics and quantum information science have attracted enormous interest in examining the quantum dynamics of open systems in various time domains and coupling strength ranges. Although the Lindblad master equation is a powerful theoretical tool to study an open quantum system under the Born-Markov approximation, such a Markov method will not be valid when the system is strongly coupled to an environment or the surrounding environment has a structured spectrum. In this case, it is inevitable to employ a non-Markovian quantum approach. However, unlike in

Y. Chen (✉) · T. Yu
Department of Physics and Engineering Physics, Stevens Institute of Technology, Castle Point on Hudson, Hoboken, NJ 07030, USA
e-mail: yusui.chen@stevens.edu

T. Yu
e-mail: ting.yu@stevens.edu

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the case of the standard Markov regimes, deriving the evolution equation that governs the density operator for a non-Markovian open quantum system is a long outstanding open problem. The recently developed non-Markovian quantum state diffusion (QSD) approach [1] offers an alternative way of solving the non-Markovian open quantum systems. However, from a more fundamental point of view, particularly in conjunction with the investigation of quantum decoherence and non-equilibrium quantum transport, a non-Markovian master equation approach that can be applied to both strong coupling regimes and structured medium is also highly desirable.

In this paper, we report a systematic theoretical approach that can be implemented easily for realistic quantum systems such as multiple-qubit systems [2]. The paper is organized as follows. In Sect. 25.2, we describe the principle ideas of establishing stochastic Schrödinger equations for a generic open quantum system coupled to a bosonic bath. We further present our recent work on developing a systematic non-Markovian master equation based on the stochastic non-Markovian QSD approach in Sect. 25.3. In Sect. 25.4, as examples, we study both two-qubit and three-qubit systems analytically with our new master equation approach. Some technical details are left to appendices.

25.2 Non-Markovian Quantum-State Diffusion Approach

The model under consideration is a generic open quantum system linearly coupled to a zero-temperature bosonic environment. The total Hamiltonian may be written as (setting $\hbar = 1$) [3–5]:

$$\begin{aligned} H_{tot} &= H_{sys} + H_{int} + H_{env} \\ &= H_{sys} + \sum_k \left(g_k L b_k^\dagger + g_k^* L^\dagger b_k \right) + \sum_k \omega_k b_k^\dagger b_k, \end{aligned} \tag{25.1}$$

where H_{sys} is the Hamiltonian of an arbitrary quantum system of interest, such as spins, atoms, quantum harmonic oscillators, cavities etc. The operator L is an arbitrary system operator, describing the coupling between the system of interest and its environment. $b_k(b_k^\dagger)$ is the bosonic annihilation (creation) operator of kth mode in the environment, satisfying the usual commutation relations for bosons, $[b_k, b_{k'}^\dagger] = \delta_{k,k'}$ and $[b_k, b_{k'}] = [b_k^\dagger, b_{k'}^\dagger] = 0$.

In the interaction picture with respect to the environment, the total Hamiltonian can be rewritten as (the rest of this paper is discussed in the interaction picture),

$$H_{tot} = H_{sys} + \sum_k \left(g_k L b_k^\dagger e^{i\omega_k t} + g_k^* L^\dagger b_k e^{-i\omega_k t} \right). \tag{25.2}$$

With the total Hamiltonian, the evolution for the state of the total system $|\psi_{tot}(t)\rangle$ is governed by the standard Schrödinger equation,

$$\partial_t|\psi_{tot}(t)\rangle = -i\left[H_{sys} + \sum_k \left(g_k L b_k^\dagger e^{i\omega_k t} + g_k^* L^\dagger b_k e^{-i\omega_k t}\right)\right]|\psi_{tot}(t)\rangle. \qquad (25.3)$$

For a real-world problem, solving above Schrödinger equation in a non-Markovian regime is by no means an easy task due to the complexity arising from a large number of environmental variables and strong system-environment coupling. Therefore, it is desirable to develop a dynamical approach for dealing with a reduced density operator describing open quantum systems only. The quantum-state diffusion approach was developed based on a special choice of environmental basis consisting of a set of Bargmann coherent states $|z\rangle = |z_1\rangle \otimes |z_2\rangle \otimes \cdots \otimes |z_k\rangle \otimes \cdots$. For each mode, the Bargmann state is defined as

$$|z_k\rangle = e^{z_k b_k^\dagger}|0\rangle,$$

satisfying the following properties,

$$\begin{aligned} b_k|z_k\rangle &= z_k|z_k\rangle, \\ b_k^\dagger|z_k\rangle &= \frac{\partial}{\partial z_k}|z_k\rangle. \end{aligned}$$

It should be noted that the Bargmann states completeness identity is given by,

$$I = \int \frac{d^2z}{\pi} e^{-|z|^2}|z\rangle\langle z|,$$

where $d^2z = d^2z_1 d^2z_2 \cdots$. Then the state $|\psi_{tot}(t)\rangle$ for the combined total system can be expanded as,

$$\begin{aligned} |\psi_{tot}(t)\rangle &= \int \frac{d^2z}{\pi} e^{-|z|^2}|z\rangle\langle z|\psi_{tot}(t)\rangle \\ &= \int \frac{d^2z}{\pi} e^{-|z|^2}|\psi_t(z^*)\rangle \otimes |z\rangle, \end{aligned} \qquad (25.4)$$

where

$$|\psi_t(z^*)\rangle = \langle z|\psi_{tot}\rangle.$$

Note that $|\psi_t(z^*)\rangle$ is a pure state in the system's Hilbert space, containing the complex variables z^* that will be interpreted as complex Gaussian random variables. For reasons to be explained later, $|\psi_t(z^*)\rangle$ is called a quantum trajectory

[1, 6]. Remarkably, the reduced density operator ρ_t at time point t for the system of interest can be recovered by the quantum pure state as shown below. By definition, the reduced density operator ρ_t may be obtained from ρ_{tot} by performing the partial trace over the environmental variables. For this purpose, we choose Bargmann coherent states as our basis,

$$\begin{aligned}\rho_t &= tr_{env}(\rho_{tot}) \\ &= \int \frac{d^2 z}{\pi} e^{-|z|^2} \langle z|\psi_{tot}\rangle\langle\psi_{tot}|z\rangle \\ &= \int \frac{d^2 z}{\pi} e^{-|z|^2} |\psi_t(z^*)\rangle\langle\psi_t(z)| \\ &= \mathscr{M}(|\psi_t(z^*)\rangle\langle\psi_t(z)|),\end{aligned} \tag{25.5}$$

where the symbol

$$\mathscr{M}(\cdot) = \int \frac{d^2 z}{\pi} e^{-|z|^2} (\cdot) \tag{25.6}$$

stands for the statistical average over the random variables z^* [1, 6, 7].

From (25.3), one can derive a stochastic differential equation for a quantum trajectory when the environmental bath is in a vacuum state [1],

$$\begin{aligned}\partial_t|\psi_t(z^*)\rangle &= -i\langle z|\left[H_{sys} + \sum_k \left(g_k L b_k^\dagger e^{i\omega_k t} + h.c.\right)\right]|\psi_{tot}(t)\rangle \\ &= \left[-iH_{sys} + Lz_t^* - iL^\dagger \sum_k g_k^* \frac{\partial}{\partial z_k^*} e^{-i\omega_k t}\right]|\psi_t(z^*)\rangle,\end{aligned} \tag{25.7}$$

where

$$z_t^* = -i\sum_k g_k z_k^* e^{i\omega_k t} \tag{25.8}$$

is a complex Gaussian process.

In a more general situation where the environment is in a thermal equilibrium state

$$\rho_{env}(0) = \frac{e^{-\beta \sum_k \omega_k b_k^\dagger b_k}}{Z},$$

where $\beta = \frac{1}{k_B T}$ and Z is the partition function $Z = tr(e^{-\beta\sum_k \omega_k b_k^\dagger b_k})$, the bath correlation function can be written in the following form

$$\alpha(t,s) = \sum_k |g_k|^2 \left[\coth \frac{\omega_k}{2k_B T} \cos \omega_k (t-s) - i \sin \omega_k (t-s) \right].$$

For the zero temperature case, the correlation function reduces to

$$\alpha(t,s)|_{T=0} = \sum_k |g_k|^2 e^{-i\omega_k (t-s)}. \tag{25.9}$$

It is interesting to note that the stochastic process defined in (25.8) satisfies,

$$\begin{aligned} \mathscr{M}(z_t) &= 0, \\ \mathscr{M}(z_t z_s) &= 0, \\ \mathscr{M}(z_t^* z_s) &= \alpha(t,s). \end{aligned} \tag{25.10}$$

Equation (25.10) shows that z_t^* typically represents a non-Markovian Gaussian process characterised by the correlation $\alpha(t,s)$. Taking the Lorenz spectrum as an example,

$$J(\omega) = \frac{\Gamma}{2\pi} \frac{1}{(\omega - \omega_s + \Omega_c)^2 + \gamma^2},$$

we can explicitly show that the correlation function takes a very simple form,

$$\alpha(t,s) = \frac{\Gamma\gamma}{2} e^{(-\gamma + i\Omega_c)|t-s|}, \tag{25.11}$$

which is commonly called the Ornstein-Uhlenbeck type correlation function. Ω_c represents the central frequency of the environment and $\frac{1}{\gamma}$ is the correlation-time of the environment. When the parameter $\gamma \to \infty$, the Ornstein-Uhlenbeck correlation function recovers the well-known Markov approximation described by a Dirac delta function,

$$\alpha(t,s) \approx \Gamma \delta(t,s).$$

In (25.7), the term $\frac{\partial}{\partial z_k^*} |\psi_t(z^*)\rangle$ can be cast as a functional derivative by using the chain rule,

$$\begin{aligned} -i \sum_k g_k^* L^\dagger e^{-i\omega_k t} \frac{\partial}{\partial z_k^*} |\psi_t(z^*)\rangle &= -i \sum_k g_k^* L^\dagger e^{-i\omega_k t} \int_0^t ds \frac{\partial z_s^*}{\partial z_k^*} \frac{\delta}{\delta z_s^*} |\psi_t(z^*)\rangle \\ &= -L^\dagger \int_0^t ds \alpha(t,s) \frac{\delta}{\delta z_s^*} |\psi_t(z^*)\rangle. \end{aligned}$$

By defining the O operator,

$$O(t,s,z^*)|\psi_t(z^*)\rangle = \frac{\delta}{\delta z_s^*}|\psi_t(z^*)\rangle, \tag{25.12}$$

the non-Markovian quantum-state diffusion (QSD) equation driven by the complex Gaussian process z_t^* is written as,

$$\partial_t|\psi_t(z^*)\rangle = \left(-iH_{sys} + Lz_t^* - L^\dagger \bar{O}(t,z^*)\right)|\psi_t(z^*)\rangle, \tag{25.13}$$

where $\bar{O}(t,z^*) = \int_0^t ds\alpha(t,s)O(t,s,z^*)$.

The exact non-Markovian QSD equations are generic for open quantum system models represented by (25.1). Note that these non-Markovian stochastic equations are derived from the generic microscopic Hamiltonian (25.1) or (25.2) without any approximation. For practical numerical simulations, it is useful to recast the QSD equation into a time convolutionless form by introducing a time-local operator O. The dynamical equation of the O operator can be determined by its consistency condition,

$$\frac{\partial}{\partial t}\frac{\delta}{\delta z_s^*}|\psi_t(z^*)\rangle \equiv \frac{\delta}{\delta z_s^*}\frac{\partial}{\partial t}|\psi_t(z^*)\rangle.$$

Putting the definition of O operator (25.12) and the QSD (25.13) into above equation, the dynamical equation of O operator is given by,

$$\partial_t O(t,s,z^*) = [-iH_{sys} + Lz_t^* - L^\dagger \bar{O}(t,z^*), O(t,s,z^*)] - L^\dagger \frac{\delta \bar{O}}{\delta z_s^*}. \tag{25.14}$$

with the initial condition

$$O(t,s=t,z^*) = L. \tag{25.15}$$

For many interesting models, such as dephasing models [8], multiple-qubit dissipative systems [9–12], and quantum Brownian motion [13], the exact non-Markovian QSD equations have been established [5, 14–17]. Consequently, one can study the non-Markovian behaviors of quantum decoherence and quantum entanglement, based on the numerically recovered reduced density operator ρ_t. However, from a more fundamental point of view, it is known that the corresponding non-Markovian master equations are very useful in describing quantum dissipative dynamics, quantum transport processes, and quantum decoherence. Therefore, it is of great interest to establish a generic relation between the stochastic QSD equations and their master equation counterparts.

25.3 Non-Markovian Master Equation Approach

After discussing the non-Markovian QSD approach, we will study the relationship between the non-Markovian QSD and master equation approaches in this section. As a fundamental tool, the master equation governs the evolution of the reduced density operator for an open quantum system. However, deriving a systematic non-Markovian master equation for a generic open quantum system is a rather difficult problem. Up to now, exact master equations are available only for some specific models, such as the dephasing model, qubit dissipative model, and Brownian motion model [5, 7, 13–26]. Traditionally in quantum optics, in the case of weak coupling and broadband approximation, one can adequately describe the dynamics of atoms coupled to a quantized radiation field by a Lindblad master equation [27],

$$\partial_t \rho_t = [-iH_{sys}, \rho_t] - \frac{\Gamma}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L - 2L\rho_t L^\dagger), \tag{25.16}$$

where ρ_t is the reduce density operator of the system of interest, L is the Lindblad operator and Γ represents a decay rate. However, when the Born-Markov approximation ceases to be valid as shown in many cases involving strong couplings and structured spectrum distributions, non-Markovian dynamics has to be invoked. It is shown that the non-Markovian dynamics can bring new interesting physical phenomena, such as a regeneration of quantum entanglement, slow quantum coherence decay and so on. In this section, we show a systematic way of deriving the non-Markovian master equations from stochastic QSD equations.

As shown in (25.5), the reduced density matrix ρ_t can be formally recovered by taking the statistical average over all the quantum trajectories,

$$\rho_t = \mathscr{M}[|\psi_t(z^*)\rangle\langle\psi_t(z)|].$$

From this starting point, we can write down the formal master equation as,

$$\partial_t \rho_t = [-iH_{sys}, \rho_t] + L\mathscr{M}[z_t^* P_t] - L^\dagger \mathscr{M}[\bar{O}(t, z^*)P_t] + \mathscr{M}[z_t P_t]L^\dagger - \mathscr{M}[P_t \bar{O}^\dagger(t, z)]L, \tag{25.17}$$

where P_t is the stochastic projection operator $P_t(z, z^*) = |\psi_t(z^*)\rangle\langle\psi_t(z)|$.

By applying the Novikov's theorem [8],

$$\mathscr{M}[z_t^* P_t] = \int_0^t ds \mathscr{M}[z_t^* z_s]\mathscr{M}[\frac{\delta P_t}{\delta z_s}],$$

it is easy to obtain the following results,

$$\mathcal{M}[z_t^* P_t] = \mathcal{M}[P_t \bar{O}^\dagger],$$
$$\mathcal{M}[z_t P_t] = \mathcal{M}[\bar{O} P_t]. \tag{25.18}$$

The detailed proof of the above results can be found in the Appendix 1.

Therefore, the formal master equations can be written as

$$\partial_t \rho_t = [-iH_{sys}, \rho_t] + [L, R(t)] - [L^\dagger, R^\dagger(t)]. \tag{25.19}$$

where,

$$R(t) = \mathcal{M}(P_t \bar{O}^\dagger).$$

As a note, we point out that non-Markovian master equations may provide a possibility to find an exact analytical solution. Even in numerical simulations, in some cases, such as small quantum systems, a master equation can significantly reduce computational complexity. Generally, the O operator contains noise terms, therefore, the term $\mathcal{M}(\bar{O}P_t)$ is still hard to derive analytically.

Example Here we consider the one qubit dissipative model as an example to show how to use Novikov's theorem to derive an exact master equation. The total Hamiltonian in this case is given by [1, 8],

$$H_{tot} = \frac{\omega}{2}\sigma_z + \sigma_- \sum_k g_k b_k^\dagger e^{i\omega_k t} + \sigma_+ \sum_k g_k^* b_k e^{-i\omega_k t}.$$

Then, the non-Markovian QSD (25.13) can be explicitly written as,

$$\partial_t |\psi_t(z^*)\rangle = (-i\frac{\omega}{2}\sigma_z + \sigma_- z_t^* - \sigma_+ \bar{O})|\psi_t(z^*)\rangle. \tag{25.20}$$

And the O operator takes the form of

$$O(t,s) = f(t,s)\sigma_-, \tag{25.21}$$

where the coefficient function $f(t,s)$ satisfies the initial condition $f(t,t) = 1$ and it obeys the equation of motion,

$$\partial_t f(t,s) = i\omega f + Ff,$$
$$F(t) = \int_0^t ds \alpha(t,s) f(t,s).$$

Here we choose the Ornstein-Uhlenbeck type correlation function (25.11) as an example, such that the coefficient function $F(t)$ satisfies

$$\frac{d}{dt}F(t) = \frac{\Gamma\gamma}{2} - \gamma F + i\omega F + F^2,$$
$$F(0) = 0.$$

Using Novikov's theorem (25.18), we have

$$\mathscr{M}(\bar{O}P_t) = F(t)\sigma_-\rho_t,$$
$$\mathscr{M}(P_t\bar{O}^\dagger) = F^*(t)\rho_t\sigma_+.$$

Then the exact master equation can be shown explicitly as

$$\partial_t\rho_t = [-i\frac{\omega}{2}\sigma_z, \rho_t] - (F\sigma_+\sigma_-\rho_t + F^*\rho_t\sigma_+\sigma_- - (F+F^*)\sigma_-\rho_t\sigma_+). \tag{25.22}$$

Next, we check its Markov limit: writing the correlation function in the form

$$\alpha(t,s) = \Gamma\delta(t,s), \tag{25.23}$$

then $F(t)$ can be calculated as

$$F(t) = \int_0^t ds\Gamma\delta(t,s)f(t,s) = \frac{\Gamma}{2}.$$

The master equation in the Markov limit is easily obtained from (25.22),

$$\partial_t\rho_t = \left[-i\frac{\omega}{2}\sigma_z, \rho_t\right] - \frac{\Gamma}{2}(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_- - 2\sigma_-\rho_t\sigma_+), \tag{25.24}$$

which clearly takes the standard Lindblad form.

25.4 Multiple-Qubit Systems

In this section, we discuss a multiple-qubit system coupled to a common bosonic environment. The multiple-qubit model is of interest in quantum information as it represents a quantum memory realised by two-level systems such as spins or atoms [28–34]. Studies of dissipation and decoherence for multiple qubit systems are useful to understand quantum decoherence control and quantum disentanglement processes. Such studies can help us to develop new theoretical and experimental strategies to control quantum decoherence [35–37]. Here, we consider a generic N-qubit model,

$$H_{tot} = H_{sys} + L\sum_k g_k b_k^{\dagger} e^{i\omega_k t} + L^{\dagger}\sum_k g_k^* b_k e^{-i\omega_k t},$$
$$H_{sys} = \sum_j \frac{\omega_j}{2}\sigma_z^j + J_{xy}\sum_j \left(\sigma_x^j \sigma_x^{j+1} + \sigma_y^j \sigma_y^{j+1}\right),$$

where $L = \sum_j \kappa_j \sigma_-^j$ is the dissipative coupling operator of the system, κ_j is the coupling constant for j^{th} qubit. The non-Markovian QSD equation is written as

$$\partial_t |\psi_t(z^*)\rangle = \left(-iH_{sys} + Lz_t^* - L^{\dagger}\bar{O}\right)|\psi_t(z^*)\rangle, \tag{25.25}$$

where O operator is determined by the following equation,

$$\partial_t O(t,s,z^*) = [-iH_{sys} + Lz_t^* - L^{\dagger}\bar{O}(t), O(t,s)] - L^{\dagger}\frac{\delta\bar{O}(t)}{\delta z_s^*}, \tag{25.26}$$

together with the initial condition $O(t,s=t) = \sum_j \kappa_j \sigma_-^j$.

Differing from the previous simple example, O operator is no longer free of noise when the size of the system increases. In general, the O operator is typically involved with noise z^*. Note that O operator can be formally written in the functional expansion of noise [8],

$$O(t,s,z^*) = O_0(t,s) + \int_0^t ds_1 z_{s_1}^* O_1(t,s,s_1) + \int_0^t ds_1 \int_0^t ds_2 z_{s_1}^* z_{s_2}^* O_2(t,s,s_1,s_2) + \cdots, \tag{25.27}$$

where O_0 is the zeroth order, which does not contain noise z^*; also, operators O_n by definition do not contain noise. For a simple example, the one qubit case, $O = f(t,s)\sigma_-$ is a special case in which O operator only contains the O_0 term. The initial conditions for each term of the O operator are [13],

$$O_0(t,s=t) = L,$$
$$O_n(t,s=t) = 0.$$

Substituting (25.27) into (25.26), we have a set of coupled differential equations for each term O_n in the O operator (Appendix 2),

$$\begin{aligned}
\partial_t O_0(t,s) &= [-iH_{sys} - L^{\dagger}\bar{O}_0(t), O_0(t,s)] - L^{\dagger}\bar{O}_1(t,s),\\
\partial_t O_1(t,s,s_1) &= [-iH_{sys} - L^{\dagger}\bar{O}_0(t), O_1(t,s,s_1)] - [L^{\dagger}\bar{O}_1(t,s_1), O_0(t,s)]\\
&\quad - L^{\dagger}(\bar{O}_2(t,s_1,s) + \bar{O}_2(t,s,s_1)),\\
etc.,
\end{aligned} \tag{25.28}$$

together with the boundary conditions

$$\begin{aligned}
O_1(t,s,t) &= [L, O_0(t,s)],\\
O_2(t,s,t,s_1) + O_2(t,s,s_1,t) &= [L, O_1(t,s,s_1)],\\
&etc.
\end{aligned}$$

As we have shown in (25.19), explicitly finding $R(t)$ is the key to determine the exact master equation. In the next section, we will exhibit the detail of deriving $R(t)$ for some important qubit systems.

25.4.1 Two-Qubit Systems

For simplicity, we take the two-qubit system as our first example to show the details of our analytical derivation. The two-qubit system has generated enormous interest due to its relevance in quantum computing and quantum information. For example, the entanglement measure for a qubit system takes a particular simple form for the two-qubit system known as concurrence [38]. The Hamiltonian for the two-qubit model is given by,

$$\begin{aligned}
H_{tot} &= H_{sys} + L\sum_k g_k b_k^{\dagger} e^{i\omega_k t} + L^{\dagger}\sum_k g_k^* b_k e^{-i\omega_k t},\\
H_{sys} &= \frac{\omega_1}{2}\sigma_z^1 + \frac{\omega_2}{2}\sigma_z^2 + J_{xy}\left(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2\right),\\
L &= \kappa_1\sigma_-^1 + \kappa_2\sigma_-^2.
\end{aligned}$$

As discussed above, the non-Markovian QSD equation is given by,

$$\partial_t|\psi_t(z^*)\rangle = \left(-iH_{sys} + Lz_t^* - L^{\dagger}\bar{O}\right)|\psi_t(z^*)\rangle, \tag{25.29}$$

where the O operator can be written as

$$O(t,s,z^*) = O_0(t,s) + \int_0^t ds_1 z_{s_1}^* O_1(t,s,s_1), \tag{25.30}$$

where,

$$O_0(t,s) = f_1(t,s)\sigma_-^1 + f_2(t,s)\sigma_-^2 + f_3(t,s)\sigma_z^1\sigma_-^2 + f_4(t,s)\sigma_-^1\sigma_z^2, \tag{25.31}$$

$$O_1(t,s,s_1) = f_5(t,s,s_1)\sigma_-^1\sigma_-^2. \tag{25.32}$$

Inserting the explicit form of O operator into the equation of motion (25.26),

$$\partial_t O_0 = [-iH_{sys} - L^\dagger \bar{O}_0,\, O_0] - L^\dagger \int_0^t d\tau \alpha(t,\tau) f_5(t,\tau,s)\sigma_-^1\sigma_-^2, \tag{25.33}$$

$$\partial_t O_1 = [-iH_{sys},\, O_1] - [L^\dagger \bar{O}_0,\, O_1] - [L^\dagger \bar{O}_1,\, O_0], \tag{25.34}$$

we have the evolution equations for the coefficient functions as

$$\begin{aligned}\partial_t f_1 = {} & i\omega_1 f_1 - 2iJ_{xy} f_3 + (\kappa_1 F_1 + \kappa_2 F_3) f_1 + \kappa_2 (F_4 - F_1) f_3 \\ & + (\kappa_1 F_4 + \kappa_2 F_3) f_4 - \frac{\kappa_2}{2} F_5,\end{aligned} \tag{25.35}$$

$$\begin{aligned}\partial_t f_2 = {} & i\omega_2 f_2 - 2iJ_{xy} f_4 + (\kappa_1 F_4 + \kappa_2 F_2) f_2 + (\kappa_1 F_4 + \kappa_2 F_3) f_3 \\ & + \kappa_1 (F_3 - F_2) f_4 - \frac{\kappa_1}{2} F_5,\end{aligned} \tag{25.36}$$

$$\begin{aligned}\partial_t f_3 = {} & i\omega_2 f_3 - 2iJ_{xy} f_1 + \kappa_1 (F_3 - F_2) f_1 + (\kappa_1 F_4 + \kappa_2 F_3) f_2 \\ & + (\kappa_1 F_4 + \kappa_2 F_2) f_3 - \frac{\kappa_1}{2} F_5,\end{aligned} \tag{25.37}$$

$$\begin{aligned}\partial_t f_4 = {} & i\omega_1 f_4 - 2iJ_{xy} f_2 + (\kappa_1 F_4 + \kappa_2 F_3) f_1 + \kappa_2 (F_4 - F_1) f_2 \\ & + (\kappa_1 F_1 + \kappa_2 F_3) f_4 - \frac{\kappa_2}{2} F_5,\end{aligned} \tag{25.38}$$

$$\begin{aligned}\partial_t f_5 = {} & i(\omega_1 + \omega_2) f_5 + (\kappa_1 F_1 + \kappa_1 F_4 + \kappa_2 F_2 + \kappa_2 F_3) f_5 \\ & + (\kappa_1 f_1 - \kappa_1 f_4 + \kappa_2 f_2 - \kappa_2 f_3) F_5,\end{aligned} \tag{25.39}$$

where $F_j(t) = \int_0^t d\tau \alpha(t,\tau) f_j(t,\tau)$ $(j = 1,2,3,4)$ and $F_5(t,s) = \int_0^t d\tau \alpha(t,\tau) f_5(t,\tau,s)$. Based on the previous discussion, we have the initial conditions as

$$f_1(t,t) = \kappa_1,\, f_2(t,t) = \kappa_2, \tag{25.40}$$

$$f_3(t,t) = 0,\;\; f_4(t,t) = 0, \tag{25.41}$$

$$f_5(t,t,s_1) = 0, \tag{25.42}$$

and the boundary condition,

$$f_5(t,s,t) = 2(\kappa_1 f_3(t,s) + \kappa_2 f_4(t,s)). \tag{25.43}$$

For this two-qubit model, $R(t) = \mathscr{M}(P_t \bar{O}^\dagger)$ can be evaluated explicitly. By the ansatz of O operator,

$$\begin{aligned} R(t) &= \mathscr{M}(P_t \bar{O}^\dagger) \\ &= \mathscr{M}(P_t \bar{O}_0^\dagger + P_t \int_0^t ds_1 z_{s_1} \bar{O}_1^\dagger(t,s_1)). \end{aligned}$$

Since both O_0 and O_1 are free of noise, therefore, we have,

$$R(t) = \rho_t \bar{O}_0^\dagger + \int_0^t ds_1 \mathscr{M}(z_{s_1} P_t) \bar{O}_1^\dagger(t,s_1). \tag{25.44}$$

Applying Novikov's theorem (25.18), we obtain,

$$\begin{aligned} \mathscr{M}(z_{s_1} P_t) &= \int_0^t ds_2 \alpha(s_1,s_2) \mathscr{M}(O(t,s_2)P_t) \\ &= \int_0^t ds_2 \alpha(s_1,s_2) \left[O_0(t,s_2)\rho_t + \int_0^t ds_3 O_1(t,s_2,s_3) \mathscr{M}(z_{s_3}^* P_t) \right]. \end{aligned}$$

Repeatedly applying Novikov's theorem, we get,

$$\begin{aligned} \mathscr{M}(z_{s_3}^* P_t) &= \int_0^t ds_4 \alpha(s_3,s_4) \mathscr{M}(P_t O^\dagger(t,s_4)) \\ &= \int_0^t ds_4 \alpha(s_3,s_4) \left[\mathscr{M}(P_t O_0^\dagger(t,s_4)) + \int_0^t ds_5 \mathscr{M}(P_t z_{s_5}) O_1^\dagger(t,s_4,s_5) \right]. \end{aligned}$$

In general, repeating the Novikov theorem may generate an infinite number of terms. However, as shown below, for our two-qubit model, we can get a closed equation in a finite number of steps. Note that, if we put all the results back into $R(t)$, we have

$$R(t) = \rho_t \bar{O}_0^\dagger + \int_0^t ds_1 \int_0^t ds_2 \alpha(s_1, s_2) O_0(t, s_2) \rho_t \bar{O}_1^\dagger(t, s_1)$$
$$+ \int_0^t ds_1 \int_0^t ds_2 \alpha(s_1, s_2) \int_0^t ds_3 O_1(t, s_2, s_3) \mathscr{M}(z_{s_3}^* P_t) \bar{O}_1^\dagger(t, s_1). \quad (25.45)$$

It is easy to check that,

$$\mathscr{M}(z_{s_3}^* P_t) O_1^\dagger(t, s, s_1) = 0,$$

since

$$O_0^\dagger O_1^\dagger = 0,$$
$$O_1^\dagger O_1^\dagger = 0.$$

The two identities are called "forbidden conditions" [2], which result in a closed noise-free $R(t)$ operator,

$$R(t) = \rho_t \bar{O}_0^\dagger + \int_0^t ds_1 \int_0^t ds_2 \alpha(s_1, s_2) O_0(t, s_2) \rho_t \bar{O}_1^\dagger(t, s_1). \quad (25.46)$$

Finally, we determine the exact non-Markovian master equation for the two-qubit system in a bosonic environment. Here, we explicitly exhibit $R(t)$ with coefficient functions:

$$\begin{aligned} R^\dagger(t) = {} & (F_1 \sigma_-^1 + F_2 \sigma_-^2 + F_3 \sigma_z^1 \sigma_-^2 + F_4 \sigma_-^1 \sigma_z^2) \rho_t \\ & + \sigma_-^1 \sigma_-^2 \rho_t \left[r_1(t) \sigma_+^1 + r_2(t) \sigma_+^2 + r_3(t) \sigma_z^1 \sigma_+^2 + r_4(t) \sigma_+^1 \sigma_z^2 \right], \end{aligned} \quad (25.47)$$

where $r_j(t) = \int_0^t ds_1 \int_0^t ds_2 \alpha(s_1, s_2) f_j^*(t, s_2) F_5(t, s_1)$, $(j = 1, 2, 3, 4)$.

In Fig. 25.1, we show the dynamics of quantum entanglement in the two-qubit system. For calculational simplicity, we choose the Ornstein-Uhlenbeck type of correlation function (25.11) in our numerical simulation. Figure 25.1a shows a few single-trajectory paths, numerically simulated by the non-Markovian QSD equation. In Fig. 25.1b, we use 100-trajectory averaged (dash-dotted curve) and 1000-trajectory (dashed curve) averaged reduced density operators ρ_t to simulate the entanglement dynamics. Also we show the result simulated by using the non-Markovian master equation (solid line). The non-Markovian dynamics for 1000 quantum trajectories shows a high degree of agreement with the master equation approach.

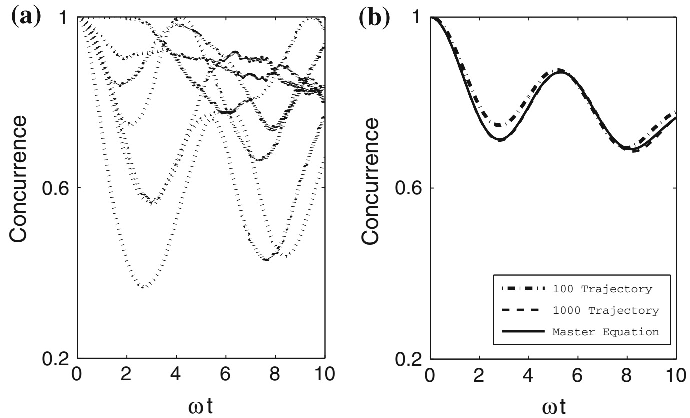

Fig. 25.1 Quantum entanglement in two-qubit system, initially prepared in a Bell state $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$. We show the results: **a** a set of single-trajectory evolution (*dashed*), and **b** 100 trajectories averaged (*dash-dotted*), 1000 trajectories averaged (*dashed*) and master equation (*solid*). The parameters are set as: $\omega_1 = \omega_2 = \omega$, $\kappa_1 = \kappa_2 = 1$, $J_{xy} = 0$ and $\gamma = 0.1$

25.4.2 Three-Qubit Systems

As another interesting example, in this section, we extend our derivation for the two-qubit system to the case of a three-qubit model. With the derived non-Markovian master equation, we study quantum decoherence and quantum disentanglement in a multiple-qubit system. Although there is no convenient computable measure of entanglement for multipartite systems, we can still investigate the entanglement transfer between two qubits in a multiple-qubit system. The total Hamiltonian for the three-qubit system (shown in Fig. 25.2) is,

$$H_{tot} = H_{sys} + L\sum_k g_k b_k^\dagger e^{i\omega_k t} + L^\dagger \sum_k g_k^* b_k e^{-i\omega_k t},$$

$$H_{sys} = \sum_{j=1}^{3} \frac{\omega_j}{2}\sigma_z^j + J_{xy}\sum_{j=1}^{2}\left(\sigma_x^j\sigma_x^{j+1} + \sigma_y^j\sigma_y^{j+1}\right),$$

where $L = \sum_{j=1}^{3}\kappa_j\sigma_-^j$ is the Lindblad operator coupling the system to the environment. The non-Markovian QSD equation in this case is given by,

$$\partial_t|\psi_t(z^*)\rangle = \left(-iH_{sys} + Lz_t^* - L^\dagger\bar{O}\right)|\psi_t(z^*)\rangle. \tag{25.48}$$

For the three-qubit dissipative model, the O operator contains up to the second-order of noise and can be written in a functional expansion as

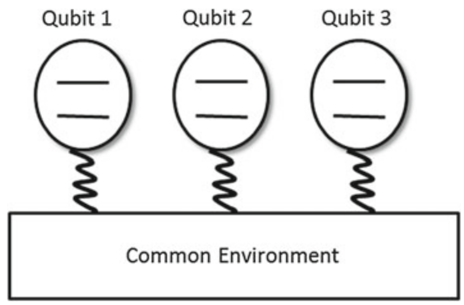

Fig. 25.2 Schematic of the 3-qubit system coupled to a common environment

$$O(t,s,z^*) = O_0(t,s) + \int_0^t ds_1 z^*_{s_1} O_1(t,s,s_1) + \int_0^t ds_1 \int_0^t ds_2 z^*_{s_1} z^*_{s_2} O_2(t,s,s_1,s_2).$$

Here, we do not explicitly show the form of O operators. However, we still have the boundary conditions and initial conditions from the O operator evolution equation. The evolution equations for O_0, O_1 and O_2 are

$$\partial_t O_0(t,s) = [-iH_{sys} - L^\dagger \bar{O}_0, O_0] - L^\dagger \bar{O}_1(t,s), \tag{25.49}$$

$$\begin{aligned} \partial_t O_1(t,s,s_1) = {} & [-iH_{sys}, O_1] - [L^\dagger \bar{O}_0, O_1] - [L^\dagger \bar{O}_1, O_0] \\ & - L^\dagger(\bar{O}_2(t,s,s_1) + \bar{O}_2(t,s_1,s)), \end{aligned} \tag{25.50}$$

$$\begin{aligned} \partial_t O_2(t,s,s_1,s_2) = {} & [-iH_{sys}, O_2] - [L^\dagger \bar{O}_0, O_2] - [L^\dagger \bar{O}_2, O_0] \\ & - [L^\dagger \bar{O}_1(t,s_1), O_1(t,s,s_2)] - [L^\dagger \bar{O}_1(t,s_2), O_1(t,s,s_1)]. \end{aligned} \tag{25.51}$$

The boundary conditions are

$$\begin{aligned} O_1(t,s,t) &= [L, O_0(t,s)], \\ O_2(t,s,t,s_1) + O_2(t,s,s_1,t) &= [L, O_1(t,s,s_1)]. \end{aligned}$$

The initial conditions are

$$\begin{aligned} O_0(t,s=t) &= L, \\ O_1(t,s=t,s_1) &= 0, \\ O_2(t,s=t,s_1,s_2) &= 0. \end{aligned}$$

And the "forbidden conditions" are

$$
\begin{aligned}
O_0 O_2 &= 0, \\
O_1 O_2 &= 0, \\
O_2 O_2 &= 0.
\end{aligned} \tag{25.52}
$$

Equations (25.49–25.51), together with their initial conditions fully determine the O operator. In order to derive the exact master equation, the last step is to evaluate $R(t) = \mathscr{M}[P_t \bar{O}^\dagger]$ in the form

$$
R(t) = \rho_t \bar{O}_0^\dagger + \int_0^t ds_1 \mathscr{M}(z_{s_1} P_t) \bar{O}_1^\dagger(t, s_1) + \int_0^t ds_1 \int_0^t ds_3 \mathscr{M}(z_{s_1} z_{s_3} P_t) \bar{O}_2^\dagger(t, s_1, s_3). \tag{25.53}
$$

Similar to the two-qubit case, employment of the Novikov theorem (25.18) and the forbidden conditions (25.52) leads to $R(t)$ of (25.53) in the form

$$
\begin{aligned}
R = {} & \rho_t \bar{O}_0^\dagger + \int_0^t ds_1 \int_0^t ds_2 \alpha(s_1, s_2) O_0(t, s_2) \rho_t \bar{O}_1^\dagger(t, s_1) \\
& + \int_0^t ds_1 \int_0^t ds_2 \int_0^t ds_3 \int_0^t ds_4 \alpha(s_1, s_2) \alpha(s_3, s_4) O_1(t, s_2, s_3) \rho_t O_0^\dagger(t, s_4) \bar{O}_1^\dagger(t, s_1) \\
& + \int_0^t ds_1 \int_0^t ds_2 \int_0^t ds_3 \int_0^t ds_4 \alpha(s_1, s_3) \alpha(s_2, s_4) O_0(t, s_3) O_0(t, s_4) \rho_t \bar{O}_2^\dagger(t, s_1, s_2) \\
& + \int_0^t ds_1 \int_0^t ds_2 \int_0^t ds_3 \int_0^t ds_4 \alpha(s_1, s_3) \alpha(s_2, s_4) O_1(t, s_3, s_4) \rho_t \bar{O}_2^\dagger(t, s_1, s_2).
\end{aligned} \tag{25.54}
$$

The detailed derivation of this can be found in Appendix 3. With the exact form of $R(t)$, the exact non-Markovian master equation may be explicitly obtained,

$$
\partial_t \rho_t = [-iH_{sys}, \rho_t] + [L, R] - [L^\dagger, R^\dagger]. \tag{25.55}
$$

It should be noted that in the above derivation, the correlation function $\alpha(t, s)$ can have an arbitrary form. Therefore our derivation of the exact master equation is completely general.

In Fig. 25.3, we plot the entanglement dynamics of a pair of qubits in the 3-qubit model with four different initial states, including a separate state and three

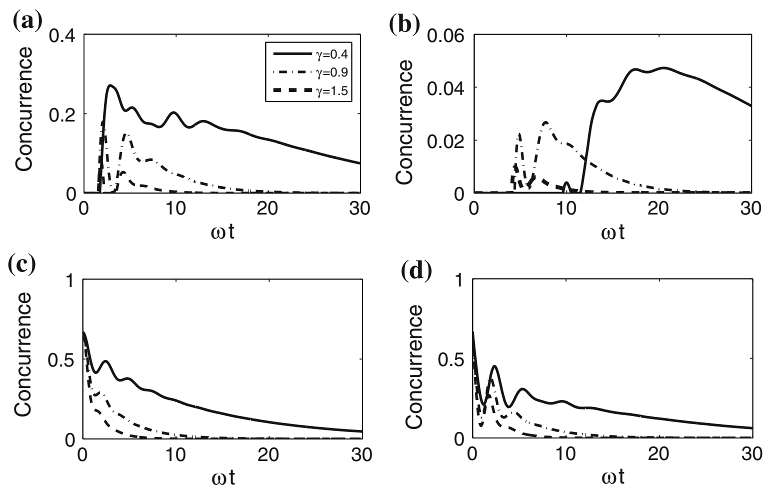

Fig. 25.3 The dynamics of entanglement between qubit 1 and qubit 2 (see Fig. 25.2) with different initial states. **a** $|111\rangle$, **b** $(|111\rangle + |000\rangle)/\sqrt{2}$, **c** $(|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$, **d** $(|110\rangle + |101\rangle + |011\rangle)/\sqrt{3}$

maximally entangled states (GHZ state and W state). Without loss of generality, the concurrence between qubit 1 and qubit 2 is studied. In Fig. 25.3a, b, since the initial three-qubit states are $|111\rangle$ and $\frac{1}{\sqrt{2}}(|111\rangle + |000\rangle)$ respectively, there is no entanglement between the qubit-pair considered. When we choose different memory times $1/\gamma$ (taking Ornstein-Uhlenbeck noise as an example again (25.11)), the degrees of the generated quantum entanglement are different. When $\gamma = 0.4$, a typical non-Markovian regime, the maximally generated entanglement is much higher than that in the case with $\gamma = 1.5$ representing the Markov limit. In Fig. 25.3c, d, the initial GHZ state of the three-qubit system is maximally entangled, and the reduced density matrices for qubits 1 and 2 are also entangled. When $\gamma = 0.4$, the early revival of entanglement in both cases is a typical non-Markovian feature.

Furthermore, we consider the entanglement transfer between two pairs of qubits. In Fig. 25.4, we prepare a Bell state for the qubit-pair 1 and 2. The idea is to observe the way entanglement transfers from qubits 1 and 2 to qubits 2 and 3. Because of the symmetry of the model, the behaviors of quantum entanglements C_{13} and C_{23} are identical. In Fig. 25.4a, c, the correlation parameter $\gamma = 0.4$ is fixed, therefore these two graphs show the short-time behavior of non-Markovian entanglement evolution. For different initial states, the speed of generating quantum entanglement is also different. In Fig. 25.4b, d, with the environment close to the Markov limit with $\gamma = 1.5$, we see that the entanglement drops to its final steady state quickly, as expected. It is interesting to note that the quantum entanglement between a pair of qubits does not actually vanish for a long time. Contrary to the two-qubit system

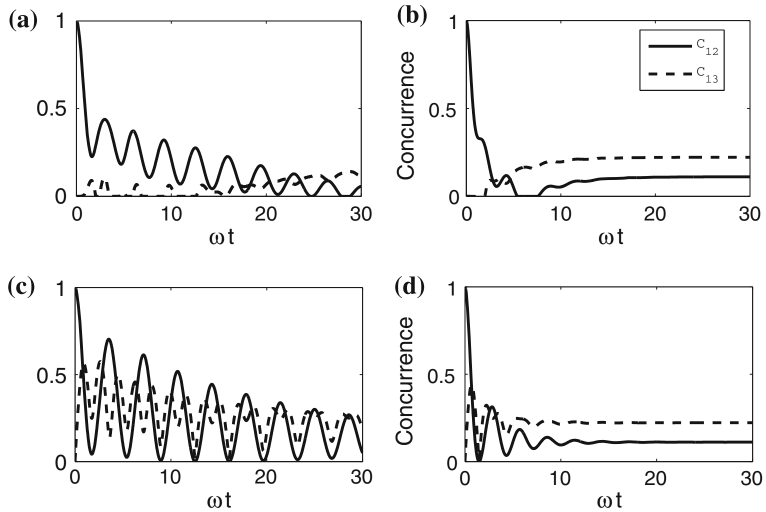

Fig. 25.4 The dynamics of quantum entanglement between three qubit-pairs C_{12} (qubit 1 and 2, *solid*) and C_{13} (qubit 1 and 3, *dashed*). *Left column* shows a non-Markovian regime with $\gamma = 0.4$. *Right column* shows a regime close to Markov limit (we choose $\gamma = 1.5$). **a** and **b** use the same initial state $(|11\rangle + |00\rangle) \otimes |0\rangle/\sqrt{2}$; while **c** and **d** use the initial state $(|10\rangle + |01\rangle) \otimes |0\rangle/\sqrt{2}$

dissipatively coupled to a bosonic environment, most two-qubit entangled states will be disentangled eventually, except for the Bell state $(|10\rangle - |01\rangle)/\sqrt{2}$, which preserves the quantum information due to the decoherence-free subspace. However, as shown in Fig. 25.4b, d, quantum entanglement can be stored in a pair of qubits robustly. This result can be naturally extended to N-qubit systems; the capacity of storing quantum information will increase as the size of quantum system is enlarged.

25.4.3 A Note on General N-Qubit Systems

We remark that the previous derivations for the two-qubit and the three-qubit systems can be extended to the more general case of N-qubit systems, with the Lindblad operator $L = \sum_j \kappa_j \sigma_-^j$. The general procedure for generalizing our results to N-qubit systems is highlighted as follows. First, we need to determine the maximum order of noise in the O operator. It is easy to prove that $L^{N+1} = 0$ for a N-qubit system, and the last term of the O operator, O_{N-1}, must be in the form of L^N. And the highest order of noise in the O operator is $N - 1$ [11]. For example, the O operator contains the first-order noise in the two-qubit model, and up to the second order of noise in the three-qubit models. Similarly, there is at most $N - 1$ order of noise for the N-qubit models. Second, we need to determine the "forbidden conditions".

The close condition for qubit is $(\sigma_-^j)^2 = 0$. One can see that if two O operator components O_j and O_k satisfy this condition $j + k > N - 2$, then $O_j O_k = 0$. Therefore, generally, one can obtain the explicit form of $R(t) = \mathscr{M}(P_t \bar{O}^\dagger)$, by calculating

$$\mathscr{M}(z_{s_1} \cdots z_{s_{2j-1}} P_t) = \int_0^t \cdots \int_0^t ds_2 \cdots ds_{2j} \left(\prod_j \alpha(s_{2j-1}, s_{2j}) \right) \mathscr{M}\left[(\prod_j \frac{\delta}{\delta z^*_{s_{2j}}}) P_t \right]. \tag{25.56}$$

Once the closed form of the $R(t)$ operator is obtained, the exact master equation is determined.

25.5 Conclusion

In this paper, based on the non-Markovian QSD approach, we analytically and numerically investigate multiple-qubit systems dissipatively coupled to a non-Markovian zero-temperature bosonic environment. We have explicitly demonstrated how to establish an exact non-Markovian master equation from the corresponding quantum state diffusion equation. Our approach is very flexible in the sense that it can be readily modified to solve many other types of models such as hybrid systems consisting of qubits, qutrits, continuous variable systems and multiple-environment systems, to name a few [39]. The time-local exact master equation approach studied in this paper represents a new advance in our investigations of non-Markovian quantum dynamics and non-equilibrium quantum dynamics. We expect that our newly developed theoretical approach will be useful in attacking many real-world problems.

Appendix 1

Here we supply a proof of the Novikov theorem. To make the proof more generic, we calculate the term $\mathscr{M}(z_\tau P_t)$, where τ and t are two independent time indexes. In this, $\mathscr{M}(z_t P_t)$ is the limit case in which $\tau = t$. By the definition of ensemble average in (25.6), we have [8]

$$\mathscr{M}(z_\tau P_t) = \int \frac{d^2 z}{\pi} e^{-|z|^2} z_\tau P_t. \tag{25.57}$$

where $|z|^2 = \sum_k |z_k|^2$ and $d^2z = d^2z_1 d^2z_2 \cdots$. With the definition of $z_\tau = i\sum_k g_k^* z_k e^{-i\omega_k\tau}$, we have

$$\mathscr{M}(z_\tau P_t) = \int \frac{d^2z_1}{\pi}\frac{d^2z_2}{\pi}\cdots\prod_n e^{-|z_n|^2}\left(i\sum_k g_k^* z_k e^{-i\omega_k\tau}\right)P_t.$$

Since all z_k are independent to each other, the above integration can be simplified as

$$\mathscr{M}(z_\tau P_t) = i\sum_k g_k^* e^{-i\omega_k\tau}\left(\prod_{n\neq k}\int\frac{d^2z_n}{\pi}e^{-|z_n|^2}\right)\int\frac{dz_k dz_k^*}{\pi}e^{-|z_k|^2}z_k P_t.$$

Integrating by parts, then we have,

$$\begin{aligned}
&\int\frac{dz_k dz_k^*}{\pi}e^{-|z_k|^2}z_k P_t\\
&=\int\frac{dz_k dz_k^*}{\pi}\left(-\frac{\partial}{\partial z_k^*}e^{-|z_k|^2}\right)P_t\\
&=\int\frac{dz_k dz_k^*}{\pi}\left[\left(-\frac{\partial}{\partial z_k^*}e^{-|z_k|^2}P_t\right)+e^{-|z_k|^2}\frac{\partial}{\partial z_k^*}P_t\right]\\
&=\int\frac{dz_k dz_k^*}{\pi}e^{-|z_k|^2}\frac{\partial}{\partial z_k^*}P_t.
\end{aligned}$$

Then

$$\mathscr{M}(z_\tau P_t) = i\sum_k g_k^* e^{-i\omega_k\tau}\int\frac{d^2z}{\pi}e^{-|z|^2}\frac{\partial}{\partial z_k^*}P_t.$$

Using the functional derivative chain rule,

$$\begin{aligned}
\mathscr{M}(z_\tau P_t) &= i\sum_k g_k^* e^{-i\omega_k\tau}\int\frac{d^2z}{\pi}e^{-|z|^2}\int_0^t ds\frac{\partial z_s^*}{\partial z_k^*}\frac{\delta}{\delta z_s^*}P_t\\
&=\int\frac{d^2z}{\pi}e^{-|z|^2}\int_0^t ds\alpha(\tau,s)O(t,s,z^*)P_t,\\
\mathscr{M}(z_\tau P_t) &= \int_0^t ds\alpha(\tau,s)\mathscr{M}[O(t,s,z^*)P_t].
\end{aligned}$$

Now we have the Novikov theorem,

$$\mathscr{M}(z_\tau P_t) = \int_0^t ds \mathscr{M}(z_\tau z_s^*)\mathscr{M}[O(t,s,z^*)P_t],$$

$$\mathscr{M}(z_\tau^* P_t) = \int_0^t ds \mathscr{M}(z_\tau^* z_s)\mathscr{M}[P_t O^\dagger(t,s,z)].$$

In the limit $\tau = t$, we obtain

$$\begin{aligned}\mathscr{M}(z_t P_t) &= \mathscr{M}(\bar{O}(t,z^*)P_t),\\ \mathscr{M}(z_t^* P_t) &= \mathscr{M}(P_t \bar{O}^\dagger(t,z)).\end{aligned} \tag{25.58}$$

Appendix 2

Inserting the expansion series of O operator (25.27) into the O operator evolution (25.26), we have

$$\partial_t O(t,s) = \partial_t O_0(t,s) + z_t^* O_1(t,s,t) + \int_0^t ds_1 z_{s_1}^* \partial_t O_1(t,s,_1) + \cdots, \tag{25.59}$$

for the left hand side. Furthermore, the right hand side of (25.26) can be expanded as

$$\begin{aligned}
&[-iH_{sys} + Lz_t^* - L^\dagger \bar{O}, O] - L^\dagger \frac{\delta \bar{O}}{\delta z_s^*}\\
&= [-iH_{sys} + Lz_t^* - L^\dagger \bar{O}_0, O_0] - L^\dagger \frac{\delta}{\delta z_s^*} \int_0^t d\tau \alpha(t,\tau) \int_0^t ds_1 z_{s_1}^* O_1(t,\tau,s_1)\\
&\quad + [-iH_{sys} + Lz_t^*, \int_0^t ds_1 z_{s_1}^* O_1] - [L^\dagger \bar{O}_0, \int_0^t ds_1 z_{s_1}^* O_1] - [L^\dagger \int_0^t ds_1 z_{s_1}^* \bar{O}_1, \bar{O}_0]\\
&\quad - L^\dagger \frac{\delta}{\delta z_s^*} \int_0^t d\tau \alpha(t,\tau) \int_0^t ds_1 \int_0^t ds_2 z_{s_1}^* z_{s_2}^* O_2(t,\tau,s_1,s_2)\\
&\quad + \cdots.
\end{aligned} \tag{25.60}$$

By the definition $\bar{O} = \int_0^t ds\alpha(t,s)O(t,s,z^*)$, we can calculate the terms

$$\frac{\delta}{\delta z_s^*}\int_0^t d\tau\alpha(t,\tau)\int_0^t ds_1 z_{s_1}^* O_1(t,\tau,s_1)$$
$$= \int_0^t d\tau\alpha(t,\tau)\int_0^t ds_1\delta(s,s_1)O_1(t,\tau,s_1)$$
$$= \int_0^t d\tau\alpha(t,\tau)O_1(t,\tau,s) = \bar{O}_1(t,s),$$

and

$$\frac{\delta}{\delta z_s^*}\int_0^t d\tau\alpha(t,\tau)\int_0^t ds_1\int_0^t ds_2 z_{s_1}^* z_{s_2}^* O_2(t,\tau,s_1,s_2)$$
$$= \int_0^t d\tau\alpha(t,\tau)\int_0^t ds_1\int_0^t ds_2 z_{s_1}^*\delta(s,s_2)O_2(t,\tau,s_1,s_2)$$
$$+ \int_0^t d\tau\alpha(t,\tau)\int_0^t ds_1\int_0^t ds_2 z_{s_2}^*\delta(s,s_1)O_2(t,\tau,s_1,s_2)$$
$$= \int_0^t d\tau\alpha(t,\tau)\int_0^t ds_1 z_{s_1}^* O_2(t,\tau,s_1,s) + \int_0^t d\tau\alpha(t,\tau)\int_0^t ds_2 z_{s_2}^* O_2(t,\tau,s,s_2)$$
$$= \int_0^t d\tau\alpha(t,\tau)\int_0^t ds_1 z_{s_1}^*(O_2(t,\tau,s_1,s) + O_2(t,\tau,s,s_1))$$
$$= \int_0^t ds_1 z_{s_1}^*(\bar{O}_2(t,s_1,s) + \bar{O}_2(t,s,s_1)).$$

Equating the two sides for each order of noise z^*, we obtain a set of dynamical equations for the O_n $(n = 1, 2, \ldots)$. For the non-noise term, we have

$$\partial_t O_0 = [-iH_{sys} - L^\dagger\bar{O}_0, O_0] - L^\dagger\bar{O}_1.$$

For the first-order noise terms, we have

$$\int_0^t ds_1 z^*_{s_1} \partial_t O_1$$
$$= \int_0^t ds_1 z^*_{s_1} \Big\{ [-iH_{sys} L^\dagger \bar{O}_0,\, O_1] - [L^\dagger \bar{O}_1,\, O_0] - L^\dagger (\bar{O}_2(t,s_1,s) + \bar{O}_2(t,s,s_1)) \Big\},$$

and the evolution equation for O_1 is obtained as

$$\partial_t O_1 = [-iH_{sys} - L^\dagger \bar{O}_0,\, O_1] - [L^\dagger \bar{O}_1,\, O_0] - L^\dagger (\bar{O}_2(t,s_1,s) + \bar{O}_2(t,s,s_1)).$$

Similarly, the set of coupled dynamical equations for all O_n can be determined sequentially. For the terms containing z^*_t, the boundary conditions can be obtained as

$$O_1(t,s,t) = [L,\, O_0(t,s)],$$
$$O_2(t,s,s_1,t) + O_2(t,s,t,s_1) = [L,\, O_1(t,s,s_1)],$$
$$etc.$$

Appendix 3

In order to explicitly derive the $R(t)$ for the three-qubit system model, we need to calculate two terms $\mathscr{M}\{z_{s_1} P_t\}$ and $\mathscr{M}\{z_{s_1} z_{s_3} P_t\}$. Since the term $\mathscr{M}\{z_{s_1} z_{s_3} P_t\}$ contains second order of noise, it can be evaluated by using Novikov's theorem twice (25.18).

$$\begin{aligned}
\mathscr{M}\{z_{s_1} P_t\} &= \int_0^t ds_2 \alpha(s_1,s_2) \mathscr{M}\{O(t,s_2) P_t\} \\
&= \int_0^t ds_2 \alpha(s_1,s_2) \left[O_0(t,s_2)\rho_t + \int_0^t ds_3 O_1(t,s_2,s_3) \mathscr{M}\{z^*_{s_3} P_t\} \right] \\
&\quad + \int_0^t ds_2 \alpha(s_1,s_2) \int_0^t ds_3 \int_0^t ds_5 O_2(t,s_2,s_3,s_5) \mathscr{M}\{z^*_{s_3} z^*_{s_5} P_t\}, \\
\mathscr{M}\{z_{s_1} z_{s_3} P_t\} &= \int_0^t ds_2 \alpha(s_1,s_2) \mathscr{M}\{z_{s_3} O(t,s_2) P_t\} \\
&= \int_0^t ds_2 \int_0^t ds_4 \alpha(s_1,s_2)\alpha(s_3,s_4) \mathscr{M}\left\{ \frac{\delta O(t,s_2)}{\delta z^*_{s_4}} P_t \right\} \\
&\quad + \int_0^t ds_2 \int_0^t ds_4 \alpha(s_1,s_2)\alpha(s_3,s_4) \mathscr{M}\{O(t,s_2) O(t,s_4) P_t\}.
\end{aligned}$$

yusui.chen@stevens.edu

After eliminating the zero terms by the "forbidden conditions", $R(t)$ can be explicitly shown as (25.54).

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