Exact non-Markovian master equations for multiple qubit systems: Quantum-trajectory approach

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A wide class of exact master equations for a multiple qubit system can be explicitly constructed by using the corresponding exact non-Markovian quantum-state diffusion equations. These exact master equations arise naturally from the quantum decoherence dynamics of qubit system as a quantum memory coupled to a collective colored noisy source. The exact master equations are also important in optimal quantum control, quantum dissipation, and quantum thermodynamics. In this paper, we show that the exact non-Markovian master equation for a dissipative N-qubit system can be derived explicitly from the statistical average of the corresponding non-Markovian quantum trajectories. We illustrated our general formulation by an explicit construction of a three-qubit system coupled to a non-Markovian bosonic environment. This multiple qubit master equation offers an accurate time evolution of quantum systems in various domains, and paves the way to investigate the memory effect of an open system in a non-Markovian regime without any approximation.

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I. INTRODUCTION

A quantum open system, its temporal evolution governed by a master equation or a stochastic Schrödinger equation, has attracted widespread interest due to its applications in various research fields such as nonequilibrium quantum dynamics, quantum control, quantum cooling, quantum decoherence, and quantum dissipation [1–12]. The quantum dynamics of an open system is commonly formulated in the system plus environment framework where the state of the open system is described by a reduced density operator. Typically, deriving the master equation governing the reduced density operator involves several important elements regarding fine details of the environment and the coupling between the system and environment. In the conventional quantum optics where the quantized radiation field is treated as an environment, the master equation for an atomic system weakly coupled to the radiation field is systematically derived, which applies the Markov approximation and takes the standard Lindblad form (setting $\hbar = 1$) [13]

$$\dot{\rho} = -i[H_{\mathrm{s}},\rho] + \sum_i (2L_i\rho L_i^\dagger - L_i^\dagger L_i\rho - \rho L_i^\dagger L_i). \quad (1)$$

Here, H_{s} is the Hamiltonian of the system of interest, and L_i are a set of system operators called Lindblad operators which couple the system to the environment.

An environment can bring about various physical phenomena to the open quantum system [12,14]. For example, in the case of two-qubit system coupled to two local bosonic baths, a Markov environment typically induces both irreversible decoherence and disentanglement [15,16]. However, the non-Markovian environment with a finite memory time can assist in regenerating quantum coherence and entanglement in the system [17–20]. Some interesting physics induced by a non-Markovian environment has been studied extensively by employing an exact or an approximate non-Markovian master equation, which has many experimental applications in quantum device, quantum information, and quantum optics [21–25].

The exact master equations provide a fundamental description to non-Markovian quantum open systems. Even if the exact stochastic Schrödinger equation is known, it is still highly desirable to derive the corresponding master equation due to its conceptual importance in understanding quantum decoherence and quantum-classical transition as well as its wide applications in quantum optics, condensed matter physics, and quantum information processing [1,12].

In the case of a non-Markovian open system, deriving a non-Markovian master equation is a notoriously difficult problem due to the lack of a systematic tool that is applicable to a generic open quantum system irrespective of the system-environment coupling strength and the environment frequency distribution [26,27]. For a quantum system coupled to a bosonic or fermionic bath, a systematic method is formulated called non-Markovian quantum-state-diffusion method (QSD) or stochastic Schrödinger equation approach [28–30]. In the non-Markovian QSD method, the quantum dynamics represented by a stochastic differential equation is driven by a Gaussian type of process z_t^*. By construction, applying the ensemble average on all possible stochastic processes, one can get the reduced density matrix of the interested system. For many models, such as multilevel atom and multiple qubit system, the exact non-Markovian dynamics have been numerically studied by using the non-Markovian QSD approach [31–36].

It is known that the Markov master equation for the open system may be derived from the corresponding stochastic unrevealing [14,37–39]. There are also some examples in the non-Markovian case where the exact master equation can be recovered from the non-Markovian QSD equation, but we need to point out that these works are derived in special conditions, such as the single-spin system [31,33,36] or quantum Brownian motion [40,41]. For a multiple qubit

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system, deriving the exact master equations from the stochastic Schrödinger equation is still an open problem. Unlike creating thousands of trajectories to recover the reduced density matrix in the QSD approach, the exact master equation can be solved deterministically, so that it can significantly improve the numerical efficiency. More importantly, the exact master equation may allow an analytical solution of non-Markovian dynamics. Such kinds of analytical evaluations give rise to useful information about quantum dissipation and decoherence in the non-Markovian regime. In this paper, we present a generic exact master equation for a multiple qubit dissipative system coupled to a non-Markovian bosonic bath. The methodology used in this paper can also be extended to a multilevel atomic system coupled to a quantized radiation field [42]. Our exact master equation provides a systematic tool in dealing with quantum coherence and optimal quantum control in a non-Markovian regime [43,44].

Our paper is organized as follows. In Sec. II, we introduce a three-qubit system and show the principle idea and the detail of analytical derivation of exact master equation for the three-qubit system. In Sec. III, we show some numerical simulation results by applying the new master equation approach. In Sec. IV, we start a general discussion on the derivation of the master equation for the N-qubit system.

II. EXACT NON-MARKOVIAN MASTER EQUATION

An N-qubit system representing a carrier of quantum information or memory is assumed to be coupled to one or more dissipative environments described by a set of harmonic oscillators. To be specific, now we consider a three-qubit model to illustrate our method of deriving the exact master equation from the non-Markovian QSD equation for a multiple qubit system. A more generic N-qubit model can be treated in a similar way. The total Hamiltonian for our three-qubit system coupled to a bosonic bath may be written as [8]

$$\begin{aligned} H_{\rm tot} &= H_{\rm s} + H_{\rm int} + H_{\rm b},\\ H_{\rm s} &= \sum_{j=1}^{3}\frac{\omega_j}{2}\sigma_z^j + J_{xy}\sum_{j=1}^{2}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y\right),\\ H_{\rm int} &= L\sum_k g_k b_k^\dagger + L^\dagger\sum_k g_k b_k,\\ H_{\rm b} &= \sum_k \omega_k b_k^\dagger b_k, \end{aligned} \tag{2}$$

where $L = \kappa_1\sigma_-^1 + \kappa_2\sigma_-^2 + \kappa_3\sigma_-^3$ is the Lindblad operator coupling system to its environment, $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ are the creation (annihilation) operators for a qubit, respectively, and $b_k(b_k^\dagger)$ is the annihilation (creation) operator of the kth mode in the bosonic environment. Note that g_k are the coupling constants between the system and its environment modes. For the case of zero temperature environment, the correlation function for the non-Markovian environment is given by $\alpha(t,s) = \sum_k |g_k|^2 e^{-i\omega_k(t-s)}$.

A. Non-Markovian QSD equation

The non-Markovian diffusive stochastic Schrödinger equation is given by [29]

$$\begin{aligned} \partial_t\psi_t(z^*) = {}& (-iH_{\rm s} + Lz_t^*)\psi_t(z^*)\\ & - L^\dagger\int_0^t ds\,\alpha(t,s)\frac{\delta}{\delta z_s^*}\psi_t(z^*), \end{aligned} \tag{3}$$

where $\psi_t(z^*)$ is the pure stochastic wave function of the three-qubit system, and $z_t^* = -i\sum_k g_k z_k^* e^{i\omega_k t}$ is the complex Gaussian stochastic process with zero mean $\mathcal{M}[z_t^*] = 0$, and correlations $\mathcal{M}[z_t^* z_s^*] = 0$ and $\mathcal{M}[z_t^* z_s] = \alpha(t,s)$. Note that $\alpha(t,s)$ is the correlation function of the bath, which determines environment memory time and dictates the transition from non-Markovian to Markov regimes. The symbol $\mathcal{M}[\ldots] = \int \frac{d^2z}{\pi}e^{-|z|^2}\ldots$ means ensemble average operation on all stochastic trajectories z_t^*.

The stochastic Schrödinger equation (3) can be transformed into a time-local form when the functional derivative of noise is replaced by an operator $O(t,s,z^*)\psi_t(z^*) = \frac{\delta}{\delta z_s^*}\psi_t(z^*)$ acting on the system's current state. In the Markov limit O operator must be the same as Lindblad operator L, therefore, the consistent initial condition for O operator is $O(t,t,z^*) = L$. By the consistency condition $\frac{\partial}{\partial t}\frac{\delta}{\delta z_s^*}\psi_t = \frac{\delta}{\delta z_s^*}\frac{\partial}{\partial t}\psi_t$, the O operator satisfies the following time-evolution equation

$$\begin{aligned} \partial_t O(t,s,z^*) = {}& [-iH_{\rm s} + Lz_t^* - L^\dagger\bar{O}(t,z^*),\, O(t,s,z^*)]\\ & - L^\dagger\frac{\delta\bar{O}(t,z^*)}{\delta z_s^*}, \end{aligned} \tag{4}$$

where $\bar{O}(t,z^*) = \int_0^t ds\,\alpha(t,s)O(t,s,z^*)$.

For the three-qubit system with dissipative coupling, the functional expansion of the O operator contains at most the two-fold noises [36]

$$\begin{aligned} O(t,s,z^*) = {}& O_0(t,s) + \int_0^t ds_1 z_{s_1}^* O_1(t,s,s_1)\\ & + \iint_0^t ds_1 ds_2 z_{s_1}^* z_{s_2}^* O_2(t,s,s_1,s_2), \end{aligned} \tag{5}$$

where $O_0(t,s), O_1(t,s,s_1), O_2(t,s,s_1,s_2)$ are three 8×8 matrices not containing any noise. One can get the evolution equations for O_i by plugging the solution (5) into Eq. (4)

$$\begin{aligned} & [-iH_{\rm s} + Lz_t^* - L^\dagger\bar{O}(t,z^*),\, O(t,s,z^*)] - L^\dagger\frac{\delta\bar{O}(t,z^*)}{\delta z_s^*}\\ & = \partial_t O_0(t,s) + \partial_t\int_0^t ds_1 z_{s_1}^* O_1(t,s,s_1)\\ & \quad + \partial_t\iint_0^t ds_1 ds_2 z_{s_1}^* z_{s_2}^* O_2(t,s,s_1,s_2). \end{aligned} \tag{6}$$

By equating the terms with the same order of noises for two sides of Eq. (6), a group of differential equations for O_0, O_1, and O_2 are given by

$$\begin{aligned} \partial_t O_0(t,s) &= [-iH_{\rm s},\, O_0] - [L^\dagger\bar{O}_0,\, O_0] + L^\dagger\bar{O}_1(t,s),\\ \partial_t O_1(t,s,s_1) &= [-iH_{\rm s},\, O_1] - [L^\dagger\bar{O}_0,\, O_1] - [L^\dagger\bar{O}_1,\, O_0]\\ &\quad - L^\dagger[\bar{O}_2(t,s,s_1) + \bar{O}_2(t,s_1,s)],\\ \partial_t O_2(t,s,s_1,s_2) &= [-iH_{\rm s},\, O_2] - [L^\dagger\bar{O}_0,\, O_2] - [L^\dagger\bar{O}_1,\, O_1]\\ &\quad - [L^\dagger\bar{O}_2,\, O_0]. \end{aligned} \tag{7}$$

Meanwhile we collect the initial conditions for these operators

$$O_1(t,s,t) = [L, O_0(t,s)],$$
$$O_2(t,s,s_1,t) + O_2(t,s,t,s_1) = [L, O_1(t,s,s_1)].$$

With the exact O operator, the time-local QSD equation is explicitly determined. Below, we will show that an exact master equation can be derived from the exact non-Markovian QSD equation.

B. Formal non-Markovian master equation

Our approach to the h andling of non-Markovian open systems defined by Eq. (2) aims to derive the exact non-Markovian master equation from the corresponding linear QSD equation (3). By design, the reduced density matrix ρ_t may be recovered from the ensemble average over all quantum trajectories. As such, the formal non-Markovian master equation can be written as

$$\partial_t \rho_t = -i[H_s, \rho_t] + L\mathcal{M}[z_t^* P_t] - L^\dagger \mathcal{M}[\bar{O} P_t] + \mathcal{M}[z_t P_t]L^\dagger - \mathcal{M}[P_t \bar{O}^\dagger]L,$$

where $\rho_t = \mathcal{M}[P_t]$ and $P_t = |\psi_t(z^*)\rangle\langle\psi_t(z)|$. Applying Novikov-type theorem $\mathcal{M}[z_t P_t] = \int_0^t ds\alpha(t,s)\mathcal{M}[O(t,s,z_t^*)P_t]$, the above formal master equation can be reorganized as the following compact form

$$\partial_t \rho_t = -i[H_s, \rho_t] + [L, \mathcal{M}[P_t \bar{O}^\dagger]] - [L^\dagger, \mathcal{M}[\bar{O} P_t]]. \quad (8)$$

From Eq. (8), it is clear that a corresponding non-Markovian equation may be obtained if one can deal with the terms containing the ensemble average $\mathcal{M}[\bar{O} P_t]$. In fact, several exact master equations derived from quantum trajectories have been worked out, including a single two-level system, harmonic oscillator model, and so on. However, up to now, deriving an exact master equation for a N-qubit system from the non-Markovian stochastic differential equation is an unsolved problem. The major purpose of this paper is explicitly to show how to accomplish this goal using the three-qubit model as a typical example. For the three-qubit model considered in this paper

$$\mathcal{M}[P_t \bar{O}^\dagger] = \rho_t \bar{O}_0^\dagger + \int_0^t ds_1 \mathcal{M}[z_{s_1} P_t]\bar{O}_1^\dagger + \iint_0^t ds_1 ds_2 \mathcal{M}[z_{s_1} z_{s_2} P_t]\bar{O}_2^\dagger.$$

Note that there are two extra terms containing the ensemble averages over noise. Moreover, when we use Novikov-type theorem for the operator mean value, we should note that the time variables for the O operator are different from that for the stochastic density matrix P_t. Hence, we have

$$\begin{aligned}
\mathcal{M}[z_{s_1} P_t] &= \int_0^t ds_2 \alpha_{1,2} \mathcal{M}[O(t,s_2)P_t],\\
\mathcal{M}[z_{s_1} z_{s_2} P_t] &= \int_0^t ds_3 \alpha_{1,3} \mathcal{M}[z_{s_2} O(t,s_3) P_t]\\
&= \iint_0^t ds_3 ds_4 \alpha_{1,3}\alpha_{2,4} \mathcal{M}[O(t,s_3)O(t,s_4)P_t]\\
&\quad + \iint_0^t ds_3 ds_4 \alpha_{1,3}\alpha_{2,4} \mathcal{M}\left[\frac{\delta O(t,s_3)}{\delta z_{s_4}^*} P_t\right],
\end{aligned} \quad (9)$$

where $\alpha_{i,j} = \alpha(s_i,s_j)$. Clearly, all the terms on the right-hand side of Eq. (9) still involve the statistical average over the noise. How to deal with these noisy terms is a crucial step in deriving the exact master equation for a multiple qubit O operator.

C. Derived exact master equation

To find the exact form of the term $\mathcal{M}[P_t \bar{O}^\dagger]$, we recall the Eq. (6), and take a careful analysis on the structure of each term in O operator expansion. In the right side of Eq. (6), the highest order of noise, coming from the term $[L^\dagger \bar{O}, O]$, goes to fourth order. While the order of noise of the right side is up to the second order. These redundant terms provide a very important observation, named as "forbidden conditions," which take the following form for the three-qubit model

$$\begin{aligned}
L O_2 = 0, \quad O O_2 = 0,\\
O_1 O_1 = 0, \quad O_1 O_0 O_0 = 0.
\end{aligned} \quad (10)$$

Now we deal with the term $\mathcal{M}[P_t \bar{O}^\dagger]$, and it is easy to eliminate several complex terms since they satisfy the "forbidden conditions." Thus the compact results in Eq. (9) are

$$\begin{aligned}
\mathcal{M}[O(t,s_2)P_t]\bar{O}_1^\dagger &= O_0(t,s_2)\rho_t \bar{O}_1^\dagger + \iint_0^t ds_3 ds_4 \alpha_{3,4} O_1(t,s_2,s_3) \times \rho_t O_0^\dagger(t,s_4)\bar{O}_1^\dagger,\\
\mathcal{M}[O(t,s_3)O(t,s_4)P_t]\bar{O}_2^\dagger &= O_0(t,s_3)O_0(t,s_4)\rho_t \bar{O}_2^\dagger\\
\mathcal{M}\left[\frac{\delta O(t,s_3)}{\delta z_{s_4}^*} P_t\right]\bar{O}_2^\dagger &= O_1(t,s_3,s_4)\rho_t \bar{O}_2^\dagger.
\end{aligned}$$

With the above results, the closed form of the ensemble average $\mathcal{M}[P_t \bar{O}^\dagger]$ can be written explicitly as

$$\begin{aligned}
R(t) &= \mathcal{M}[P_t \bar{O}^\dagger]\\
&= \rho_t \bar{O}_0^\dagger + \iint_0^t ds_1 ds_2 \alpha_{1,2} O_0(t,s_2)\rho_t \bar{O}_1^\dagger(t,s_1)\\
&\quad + \iiiint_0^t ds_1 ds_2 ds_3 ds_4 \alpha_{1,2}\alpha_{3,4} O_1(t,s_2,s_3)\\
&\quad \times \rho_t O_0^\dagger(t,s_4)\bar{O}_1^\dagger(t,s_1)\\
&\quad + \iiiint_0^t ds_1 ds_2 ds_3 ds_4 \alpha_{1,3}\alpha_{2,4}[O_0(t,s_3)O_0(t,s_4)\\
&\quad + O_1(t,s_3,s_4)]\rho_t \bar{O}_2^\dagger(t,s_1,s_2).
\end{aligned}$$

Finally, we find the exact non-Markovian master equation for three-qubit system

$$\partial_t \rho_t = -i[H_s, \rho_t] + [L, R(t)] - [L^\dagger, R^\dagger(t)]. \quad (11)$$

This exact non-Markovian master equation is the major result of this paper. In the following sections, we will apply our result to several interesting cases where the non-Markovian dynamics is studied by using our derived exact equation.

III. NUMERICAL CALCULATIONS

Below we study the non-Markovian quantum dynamics of three-qubit system. For simplicity, we use Ornstein-Uhlenbeck noise depicted by the correlation function $\alpha(t,s) = \frac{\gamma}{2}e^{-\gamma|t-s|}$. Although our master equation is universally valid for an arbitrary correction function, the advantage of choosing the Ornstein-Uhlenbeck noise is that we can control the single parameter γ to recover the Markov limit ($\gamma \to \infty$) from a non-Markov regime.

In Fig. 1, four initial states are chosen for three-qubit system. The entanglement dynamics of selected two qubits (the first and the second ones) is shown. Here we choose concurrence as the measurement of entanglement [45]. In Figs. 1(a) and 1(b), initially there is no entanglement between the two qubits. As the onset of the non-Markovian environment effects, the generation of entanglement is observed. Moreover, as shown in Fig. 1, the degree of the generated entanglement depends sensitively on the value of the parameter γ. When $\gamma = 0.4$ reprints a longer memory time, the degree of entanglement is almost five times the case with $\gamma = 1.5$, which represents a more Markovian regime. In Figs. 1(c) and 1(d), the initial state of the three-qubit system is maximally entangled for every two-qubit pair. When $\gamma = 0.4$, we observe a typical behavior in the non-Markovian regime, that is, it exhibits a stronger entanglement oscillation pattern compared to the case of $\gamma = 1.5$.

In Fig. 2, the initial state between the first and the second qubits is a Bell state, and the entanglement flow among the three qubits is studied. In Figs. 2(a) and 2(c), the system that is coupled to a non-Markovian environment exhibits a strong oscillation, and shows a symmetric pattern of the system dynamics. In particular, it shows that the entanglement shared by each pair moves forward and backward between two pairs periodically. In Figs. 2(b) and 2(d), when $\gamma = 1.5$, then the environment is close to the Markov limit, we see that the state drops to the final static state quickly as expected for a

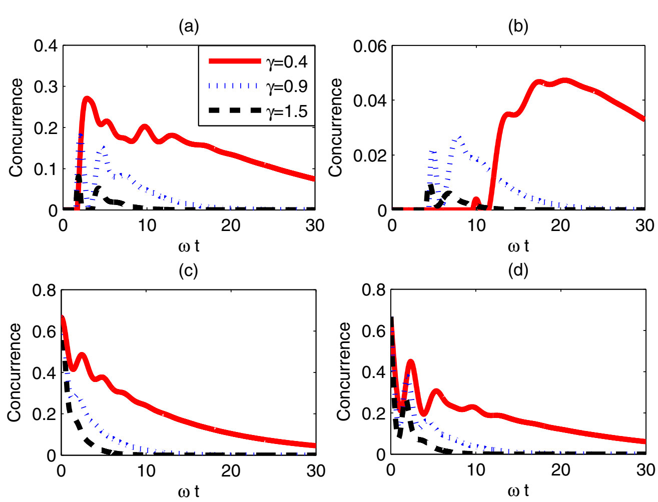

FIG. 1. (Color online) The dynamics of concurrence between the first and the second qubits as a function of ωt with different initial states. (a) $|111\rangle$, (b) $(|111\rangle + |000\rangle)/\sqrt{2}$, (c) $(|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$, and (d) $(|110\rangle + |101\rangle + |011\rangle)/\sqrt{3}$.

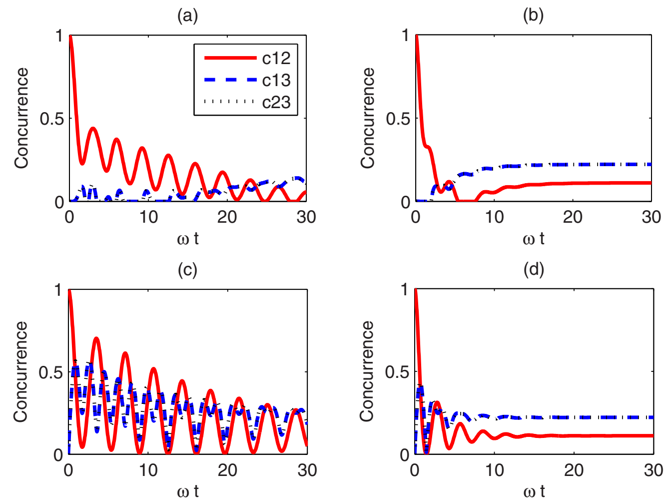

FIG. 2. (Color online) The dynamics of entanglement flow among three qubits $c12$ (red solid line), $c13$ (blue dashed line), and $c23$ (black dotted line). Left column shows a non-Markovian regime with $\gamma = 0.4$. Right column shows a regime close to Markov limit (we choose $\gamma = 1.5$). Panels (a) and (b) use the same initial state $(|11\rangle + |00\rangle) \otimes |0\rangle/\sqrt{2}$; while panels (c) and (d) use the initial state $(|10\rangle + |01\rangle) \otimes |0\rangle/\sqrt{2}$.

Markov regime. It is interesting to note that the entanglement that is present in one pair initially, will diffuse into three pairs eventually. As we know, in a two-qubit system coupling to the environment, the Bell state $(|10\rangle - |01\rangle)/\sqrt{2}$ preserves the quantum information. However, in Figs. 2(b) and 2(d), entanglement goes to a constant after a long time evolution, which shows direct evidence that the quantum information is more easily preserved in a three-qubit system than in a two-qubit system. More importantly, the exact master equation for the multiple qubit systems will allow us to study systematically the decoherence issues when the environmental noises are colored.

IV. GENERAL DISCUSSIONS ON THE EXACT MASTER EQUATION FOR AN N-QUBIT SYSTEM

In this section, we present some general discussions on the derivation of the non-Markovian master equation for the N-qubit model, with $H_s = \sum_n \omega_n \sigma_z^{(n)}$ and $L = \sum_n \kappa_n \sigma_-^{(n)}$. Since we have shown the formal master equation as Eq. (8), which is applicable for the general case, so that the goal of deriving exact master equation is based on finding exact $R(t)$.

It is easy to prove that the $(N+1)$th order product of operator L is zero. Note that, for a matrix polynomial, it means that each term has the same order $N+1$, therefore, each term contains at least one zero factor $(\sigma_-^{(j)})^2 = 0$. On the other hand, from Eq. (4), we see that the higher order of noise in the new O operator come from the commutator relation $z_t^*[L, O]$. Combining the two conditions above, we arrive at our first conclusion that, for the N-qubit model considered in this paper, the highest order of noise is $N-1$, which means that the O operator contains only finite terms. This is the basic conclusion on which our general discussion is based. On the other hand, in Eq. (6), the order of noise for both sides should match each other. From the commutator relation $[L^\dagger \bar{O}, O]$, we have all $O_j O_k$ terms with $(j+k) \geqslant N-1$ would go to zero since the right side of Eq. (6) does not contain such order terms. Now we have a general conclusion for the "forbidden conditions"

$$O_j O_k = 0, \; (j+k \geqslant N-1). \tag{12}$$

Generally, one can get an explicit expression for $\mathcal{M}[P_t \bar{O}^\dagger]$ after applying Novikov-style theorem for multiple times. We take one term of $R(t)$ as an example

$$\begin{aligned}
&\mathcal{M}\big[z_{s_1} \dots z_{s_{2j-1}} P_t \bar{O}_j^\dagger\big] \\
&\quad = \int_0^t ds_2 \alpha_{1,2} \mathcal{M}\big[z_{s_3} \dots z_{s_{2j-1}} O(t,s_2) P_t \bar{O}_j^\dagger\big] \\
&\quad = \iint_0^t ds_2 ds_4 \alpha_{1,2}\alpha_{3,4} \\
&\quad \times \mathcal{M}\big[z_{s_5} \dots z_{s_{2j-1}} O(t,s_2) O(t,s_4) P_t \bar{O}_j^\dagger\big],
\end{aligned}$$

where $\alpha_{i,j} = \alpha(s_i,s_j)$. We can keep applying the Novikov theorem and do the iteration calculation. With the "forbidden conditions" (12), it is easy to see that the $\mathcal{M}[P_t \bar{O}^\dagger]$ will eventually become noise free. Following this procedure, we can derive the general $R(t)$ and the exact master equation for an N-qubit system coupled to a non-Markovian environment.

V. CONCLUSION

We have provided a systematic approach to deriving non-Markovian master equations from the corresponding quantum-state diffusion equations. Non-Markovian master equations for open quantum systems are of importance in describing quantum dynamics coupled to a non-Markovian environment. Our exact non-Markovian master equations provide an alternative method to handling the generic open quantum systems where the standard Markov assumption is no longer valid. For example, our derived N-qubit master equations would be useful in quantum control and quantum decoherence of quantum memory and quantum optics where the atomic systems are coupled to a high-Q cavity. Note that the quantum-state diffusion equations can be formally established for a very wide class of problems in quantum open systems, therefore, our method is likely to be useful for many other applications.

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