# Numerical methods for multiscale kinetic equations: asymptotic-preserving and hybrid methods

## Lecture 2: Asymptotic-preserving schemes (Part I)

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# Lecture 2 Outline

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2. Implicit-explicit methods
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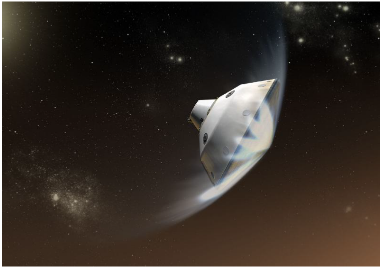
3. Application to balance laws
   - AP property
   - Stability and accuracy
   - Numerical examples
4. Concluding remarks

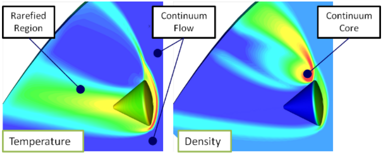
# Motivations


Intermediate Experimental Vehicle - ESA



NASA Mars Science Laboratory

- Design of spacecraft heat shields
- Hypersonic cruise vehicles
- Granular gases
- ...

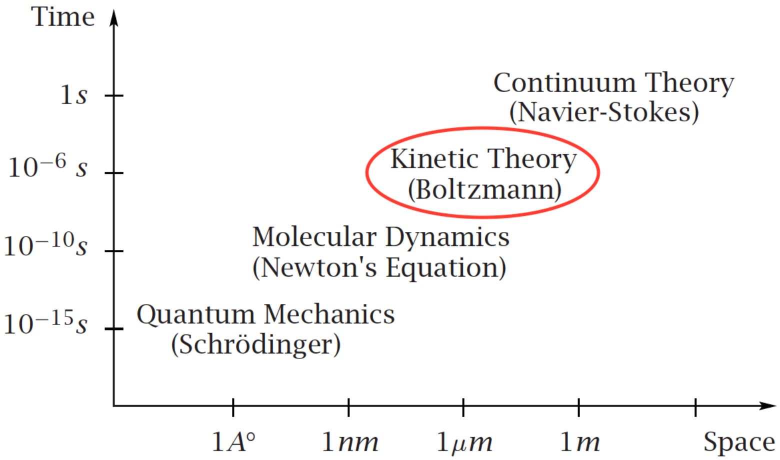



One of the most challenging phases of any space-planetary discovery mission is the stage of *hypersonic entering* into a planet's atmosphere. For the earth, reentry velocities range between $7.7$ to $15$ km/s.

- The spacecraft is exposed to various physical processes that is engendered by the synthesis of chemical kinetics, radiation physics, quantum mechanics and ablation effects with fluid dynamics.
- Due to the high altitude circumstances, the flow-regime characteristics are affected by the *breakdown of the continuum assumption*, which makes it impossible to simulate these cases with conventional CFD routines.
- Typically a model for a mixture of reacting gases is solved by DSMC (altitudes of $200$ to $85$ km) and coupled with a CFD solver for the *compressible Navier-Stokes equations* at low altitude (in the range $95$ to $65$ km) [1].

[1] G. Bird '94; J.N. Moss, C.E. Glassy, F.A. Greenez '06

# Multiscale physics



# The asymptotic-preserving (AP) property

- Numerically resolving the small scales may be computationally prohibitive and therefore one resorts on the use of some asymptotic analysis in order to derive *reduced models* which are valid in the small scales regime.
- Thus a *multi-physics* approach, that hybridizes the different models (and numerical methods) in a *domain–decomposition* framework, becomes necessary. This matching, however, is often very difficult.
- A different approach for such multiscale problems is the *asymptotic–preserving (AP)* method. The basic idea is to preserve the asymptotic procedure that lead to the reduced model in a discrete setting[2].
- The design of AP schemes needs special care for both time and space discretizations, but often, since we deal with *stiff problems*, the time discretization is more crucial.

[2] E.W. Larsen, J.E. Morel, W.F. Miller '87; F. Coron '91; S. Jin '99; L. P., G. Russo '11; P. Degond '11; G. Dimarco, L. P. '15

# A simple illustrative example

A simple prototype example of *relaxation system* is given by[3]

**Jin-Xin relaxation system**

$$P^\varepsilon : \begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + a\partial_x u = -\dfrac{1}{\varepsilon}(v - f(u)), \end{cases} \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+$$

The characteristic speeds are $\pm\sqrt{a}$. It corresponds to the original system in the *fluid scaling*: $t \to t/\varepsilon$, $x \to x/\varepsilon$. As $\varepsilon \to 0$ we get the *local equilibrium* $v = f(u)$ and we obtain

$$P^0 : \quad \partial_t u + \partial_x f(u) = 0.$$

Using the Chapman-Enskog expansion $v = f(u) + \varepsilon v_1$, under the *subcharacteristic condition* $a > |f'(u)|$, we obtain at $O(\varepsilon)$

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x \left( (a^2 - f'(u)^2) \partial_x u \right).$$

[3] S.Jin, Z.Xin '95

# The Boltzmann equation in the fluid-dynamic scaling

The density $f = f(x, v, t) \geq 0$ of particles follows

## Kinetic model

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f), \quad x \in \Omega \subset \mathbb{R}^{d_x}, v \in \mathbb{R}^3,$$

which is written in this form after the scaling $x \to x/\varepsilon$, $t \to t/\varepsilon$ where $\varepsilon > 0$ is a nondimensional parameter (*Knudsen number*) proportional to the mean free path.

- As $\varepsilon \to 0$ formally $Q(f) = 0$ which implies $f = M[f]$. Therefore, the associated moment system is closed and corresponds to the *compressible Euler equations*. This result is independent of the choice of $Q(f)$ provided it admits Maxwellian as local equilibrium functions.
- For small but non zero values of $\varepsilon$, closed evolution equations for the moments can be derived by the *Chapman-Enskog expansion* $f = M[f] + \varepsilon f_1$. This leads to the *compressible Navier-Stokes equations* as a second order approximation in $\varepsilon$ to the Boltzmann equation [4]. The choice of $Q(f)$ influences the Navier-Stokes system in terms of the *Prandtl number*.

[4] F.Golse '05

# The AP diagram

$$
\begin{tikzcd}
P^{\varepsilon} \arrow[rr, "\varepsilon \to 0"] & & P^{0} \\
& & \\
P^{\varepsilon}_{\Delta t} \arrow[uu, "\Delta t \to 0"] \arrow[rr, "\varepsilon \to 0"'] & & P^{0}_{\Delta t} \arrow[uu, "\Delta t \to 0"']
\end{tikzcd}
$$

In the diagram $P^{\varepsilon}$ is the original singular perturbation problem and $P^{\varepsilon}_{\Delta t}$ its numerical approximation characterized by a discretization parameter $\Delta t$.
The *asymptotic-preserving (AP) property* corresponds to the request that $P^{\varepsilon}_{\Delta t}$ is a good (consistent and stable) discretization of $P^{0}$ as $\varepsilon \to 0$.

# Numerical approaches

- The simplest approach is based on *splitting methods* where we solved separately the subproblems

$$\frac{\partial f}{\partial t} = \frac{1}{\varepsilon} Q(f), \qquad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0.$$

  Easy to analyze and achieve AP property, possible to use existing solvers for the simplified problems and to preserve some relevant physical properties. Main drawback: order reduction in stiff regimes.
- Different approaches to achieve high-order AP schemes
  - *IMEX Runge-Kutta methods*
  - *IMEX linear multistep methods*
  - *Exponential methods*
- All the different approaches share the difficulty of the inversion of the collision operator if evaluated implicitly.

# The Implicit-Explicit (IMEX) paradigm

Consider a systems of differential equations in the form

$$U' = \underbrace{\mathcal{F}(U)}_{\textit{non stiff terms}} + \underbrace{\mathcal{G}(U)}_{\textit{stiff terms}},$$

where $\mathcal{F}$ and $\mathcal{G}$, eventually obtained as finite-difference/element approximations of spatial derivatives, induce considerably different time scales.

- Fully explicit solvers suffer from a time step restriction induced by the stiff term $\mathcal{G}$. Since the problem is stiff as a whole implicit methods should be used.
- Fully implicit solvers, however, originate a nonlinear system of equations involving also the non stiff term $\mathcal{F}$.
- One may combine different time approximations to resolve stiff and non–stiff terms efficiently. These methods are referred to as Implicit-explicit (IMEX)[5].
- A related approach, based on Explicit exponential integrators[6], aim at solving exactly the linear stiff operator while keeping the nonlinear term explicit.

[5] U. Asher, S. Ruth, R. Spiteri, B. Wetton '95,'97; M. Carpenter, C. Kennedy '03; L. P., G. Russo '00,'05

[6] M.Hochbruck, A.Ostermann '12, L.P., G. Dimarco '11, L.P., Q. Li '15

# Numerical requirements

The combination of the implicit and explicit method should satisfy suitable order conditions. For Runge-Kutta (RK) schemes additional mixed compatibility conditions are required.

### Explicit method

- The stability region should be the largest possible.
- Monotonicity requirements

$$\|U^{n+1}\| \leq \|U^n\|, \quad \Delta t \leq \Delta t_*$$

*Strong Stability Preserving (SSP)* property[7].

### Implicit method

- Stable for stiff systems, and good damping properties.
- Computationally feasible in term of cost.

▶ The resulting scheme should be *Asymptotic Preserving (AP)* namely it should be consistent with the model reduction that occur in stiff regimes.

[7] S.Gottlieb, C-W.Shu, E.Tadmor '01, R.Spiteri, S.Ruth, '02

# The simplest IMEX-AP scheme

Consider the Jin-Xin relaxation system solved by the simple IMEX scheme

## IMEX Euler scheme

$$P_{\Delta t}^{\varepsilon}: \begin{cases} \dfrac{u^{n+1}-u^n}{\Delta t} + \partial_x v^n & = & 0, \\ \dfrac{v^{n+1}-v^n}{\Delta t} + a\partial_x u^n & = & -\dfrac{1}{\varepsilon}(v^{n+1} - f(u^{n+1})), \end{cases}$$

For small values of $\varepsilon$ we get the local equilibrium

$$v^{n+1} = f(u^{n+1})$$

which substituted into the first equation gives

$$P_{\Delta t}^{0}: \quad \frac{u^{n+1}-u^n}{\Delta t} + \partial_x f(u^n) = 0.$$

# IMEX Runge-Kutta methods[8]

## IMEX Runge-Kutta

$$
\begin{aligned}
U^{(i)} &= U^n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} \mathcal{F}(U^{(j)}) + \Delta t \sum_{j=1}^{\nu} a_{ij} \mathcal{G}(U^{(j)}), \\
U^{n+1} &= U^n + \Delta t \sum_{i=1}^{\nu} \tilde{w}_i \mathcal{F}(U^{(i)}) + \Delta t \sum_{i=1}^{\nu} w_i \mathcal{G}(U^{(i)}).
\end{aligned}
$$

$\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} = 0$, $j \geq i$ and $A = (a_{ij})$: $\nu \times \nu$ matrices and $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_\nu)^T$, $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_\nu)^T$, $c = (c_1, \ldots, c_\nu)^T$, $w = (w_1, \ldots, w_\nu)^T$.

- For *diagonally implicit schemes* (DIRK), $a_{ij} = 0$, $j > i$. They they guarantee that $\mathcal{F}$ is evaluated explicitly.
- Schemes for which $\tilde{w}_j = \tilde{a}_{\nu j}$ and $w_j = a_{\nu j}$, $j = 1, \ldots, \nu$ are called *globally stiffly accurate* (GSA).

[8] U. Ascher, S. Ruth, R. Spiteri '97, L.P., G. Russo '00

# Order conditions

- IMEX-RK schemes are a particular case of *additive Runge-Kutta (ARK)* methods [9]. Further generalization are also possible [10].
- Order conditions can be derived using a generalization of Butcher 1-trees to 2-trees.
- If $w_i = \tilde{w}_i$ and $c_i = \tilde{c}_i$ *mixed conditions* are automatically satisfied. This is not true for higher that third order accuracy

| **Order** | General case | $\tilde{w}_i = w_i$ | $\tilde{c} = c$ | $\tilde{c} = c$ and $\tilde{w}_i = w_i$ |
|---|---|---|---|---|
| **1** | 0 | 0 | 0 | 0 |
| **2** | 2 | 0 | 0 | 0 |
| **3** | 12 | 3 | 2 | 0 |
| **4** | 56 | 21 | 12 | 2 |
| **5** | 252 | 110 | 54 | 15 |
| **6** | 1128 | 528 | 218 | 78 |

[9] M. Carpenter, C. Kennedy, '03
[10] A. Sandu, M. Günther '13

# Design of IMEX-RK

Start with a $p$-order explicit SSP method and find the DIRK method that matches the order conditions with good damping properties (L-stability).

**Second order SSP IMEX-RK**

$$
\begin{aligned}
U_1 &= U^n + \gamma \Delta t \mathcal{G}(U_1) \\
U_2 &= U^n + \Delta t \mathcal{F}(U^n) + (1-2\gamma)\Delta t \mathcal{G}(U_1) + \gamma \Delta t \mathcal{G}(U_2) \\
U^{n+1} &= U^n + \frac{1}{2}\Delta t(\mathcal{F}(U^n) + \mathcal{F}(U_1)) + \frac{1}{2}\Delta t(\mathcal{G}(U_1) + \mathcal{G}(U_2)),
\end{aligned}
$$

with $\gamma = (1-\sqrt{2})/2$.

**Third order SSP IMEX-RK**

$$
\begin{aligned}
U_1 &= U^n + \gamma \Delta t \mathcal{G}(U_1) \\
U_2 &= U^n + \Delta t \mathcal{F}(U^n) + (1-2\gamma)\Delta t \mathcal{G}(U_1) + \gamma \Delta t \mathcal{G}(U_2) \\
U_3 &= U^n + \frac{1}{4}\Delta t(\mathcal{F}(U^n) + \mathcal{F}(U_1)) + (1/2-\gamma)\Delta t \mathcal{G}(U_1) + \gamma \Delta t \mathcal{G}(U_3) \\
U^{n+1} &= U^n + \frac{1}{6}\Delta t(\mathcal{F}(U^n) + \mathcal{F}(U_1) + 4\mathcal{F}(U_2)) + \frac{1}{6}\Delta t(\mathcal{G}(U_1) + \mathcal{G}(U_2) + 4\mathcal{G}(U_3)),
\end{aligned}
$$

with $\gamma = (1-\sqrt{2})/2$.

# IMEX Linear Multistep Methods[11]

## IMEX Linear Multistep

$$U^{n+1} = \sum_{j=0}^{\nu-1} a_j U^{n-j} + \Delta t \sum_{j=0}^{\nu-1} b_j \mathcal{F}(U^{n-j}) + \Delta t \sum_{j=-1}^{\nu-1} c_j \mathcal{G}(U^{n-j}),$$

with starting values $U^0, U^1, \dots, U^n$.

- The schemes are characterized by the coefficents $a = (a_0, \dots, a_{\nu-1})^T$, $b = (b_0, \dots, b_{\nu-1})^T$, $c = (c_0, \dots, c_{\nu-1})^T$ and $c_{-1} \neq 0$.
- Methods for which $c_0 = c_1 = \ldots = c_{\nu-1} = 0$ are referred to as implicit-explicit backward differentiation formula, *IMEX-BDF* in short.
- Note that *coupling conditions* in IMEX-LM can be easily satisfied (in contrast to IMEX Runge Kutta methods).
- Stability constraints usually increase with the order of the schemes. A-stable schemes have accuracy $p \leq 2$.

[11] U.Ascher, S.Ruth, B.Wetton '95, W.Hundsdorfer, S.Ruth '07

# Design of IMEX-LMM

Again we can start from an explicit SSP method and find the corresponding implicit method with good damping properties (A($\alpha$)-stability). Or we can start from an implicit method (BDF) and use the corresponding explicit scheme.

**Second order IMEX-BDF**

$$U^{n+1} = \frac{4}{3}U^n - \frac{1}{3}U^{n-1} + \frac{4}{3}\Delta t\mathcal{F}(U^n) - \frac{2}{3}\Delta t\mathcal{F}(U^{n-1}) + \frac{2}{3}\Delta t\mathcal{G}(U^{n+1}).$$

**Third order SSP IMEX-LM**

$$\begin{aligned} U^{n+1} \quad = \quad & \frac{3909}{2048}U^n - \frac{1367}{1024}U^{n-1} + \frac{873}{2048}U^{n-2} \\ + \quad & \frac{18463}{12288}\Delta t\mathcal{F}(U^n) - \frac{1271}{768}\Delta t\mathcal{F}(U^{n-1}) + \frac{8233}{12288}\Delta t\mathcal{F}(U^{n-2}) \\ + \quad & \frac{1089}{2048}\Delta t\mathcal{G}(U^{n+1}) - \frac{1139}{12288}\Delta t\mathcal{G}(U^n) - \frac{367}{6144}\Delta t\mathcal{G}(U^{n-1}) + \frac{1699}{12288}\Delta t\mathcal{G}(U^{n-2}). \end{aligned}$$

# Hyperbolic relaxation systems

Consider the case of hyperbolic relaxation systems[12]

**Hyperbolic system with relaxation (Full model)**

$$\partial_t U + \partial_x F(U) = \frac{1}{\varepsilon} R(U), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

$R : \mathbb{R}^N \to \mathbb{R}^N$ is a relaxation operator if there exists a $n \times N$ matrix $Q$ with rank$(Q) = n < N$ s.t. $QR(U) = 0 \quad \forall\, U \in \mathbb{R}^N$.
This gives $n$ conserved quantities $u = QU$ that uniquely determine a local equilibrium $U = \mathcal{E}(u)$, s.t. $R(\mathcal{E}(u)) = 0$, and satisfy

$$\partial_t (QU) + \partial_x (QF(U)) = 0.$$

As $\varepsilon \to 0 \Rightarrow R(U) = 0 \Rightarrow U = \mathcal{E}(u) \Rightarrow$ (subcharacteristic condition on $f(u)$)

**Equilibrium system (Reduced model)**

$$\partial_t u + \partial_x f(u) = 0, \qquad f(u) = QF(\mathcal{E}(u)).$$

[12] G.Chen, D.Levermore, T.P.Liu, '94

# AP property

In the case of hyperbolic system with relaxation we have the following result [13]

## Theorem (IMEX-RK)

*If $\det A \neq 0$ then in the limit $\epsilon \to 0$, the IMEX-RK scheme applied to an hyperbolic system with relaxation becomes the explicit RK scheme characterized by $(\tilde{A}, \tilde{w}, \tilde{c})$ applied to the limit system of conservation laws.*

- To satisfy $\det A \neq 0$ it is necessary that $c \neq \tilde{c}$ (Type A schemes).
- The simplification assumption $c = \tilde{c}$ is possible if the matrix $A$ can be written as (Type CK schemes)

$$\begin{pmatrix} 0 & 0 \\ a & \hat{A} \end{pmatrix}$$

  with $\det(\hat{A}) \neq 0$ where $\hat{A}$ is a $(\nu - 1) \times (\nu - 1)$ submatrix of $A$. However, the corresponding scheme may be inaccurate if the initial condition is not "well prepared" (initial layer).

[13] L.Pareschi, G.Russo, '05

# AP property

In the case of IMEX-LM methods one has the following result [14]

## Theorem (IMEX-LM)

*For arbitrary initial steps in the limit $\varepsilon \to 0$ an IMEX-BDF scheme $(w_j = 0, j = 0, \ldots, s-1)$ after $s$ time steps becomes the explicit multistep scheme characterized by $a_j, \tilde{w}_j, j = 0, \ldots, s-1$ applied to the limit system of conservation laws.*

- Note that, if the initial steps are well-prepared it can be shown that any IMEX-LM scheme satisfy the above theorem.
- Of course, both for IMEX-RK and IMEX-LM these AP results do not guarantee any stability property of the method for fixed but non zero $\varepsilon$.

[14] G. Dimarco, L.Pareschi, '15

# Stability

The $A$-stability of a IMEX scheme may be studied using the problem[15]

**Test problem**

$$u' = \lambda u + \mu u, \quad u(0) = 1, \quad \lambda, \mu \in \mathbb{C}.$$

This test problem characterizes the stability properties for linear systems

$$U' = A\,U + B\,U, \quad U(0) = U_0$$

only if $A$ and $B$ are normal, commuting matrices. In general the two matrices do not share the same eigenvectors, and can not be diagonalized simultaneously. This makes the stability analysis for systems very difficult.

▶ Recent nonlinear stability and contractivity results by Higueras et al. '04-'09, Sandu and Günther '13, L.P. and Dimarco '13.

[15] U.Asher, S.Ruuth, R.Spiteri '97, J.Frank, W.Hundsdorfer, J.Verwer '97, L.P., G.Russo '00

# Accuracy

Simple uniform error estimates can be based on the following argument.
If $P^\varepsilon_{\Delta t}$ is a $p$-order approximation of $P^\varepsilon$ then classical analysis gives

$$E_1 = \|P^\varepsilon_{\Delta t} - P^\varepsilon\| = O(\Delta t^p/\varepsilon^r), \quad 1 \leq r \leq p.$$

The AP-property typically gives

$$\|P^\varepsilon_{\Delta t} - P^0_{\Delta t}\| = O(\varepsilon), \quad \|P^0_{\Delta t} - P^0\| = O(\Delta t^p).$$

From the previous estimates one gets immediately

$$E_2 = \|P^\varepsilon_{\Delta t} - P^\varepsilon\| = O(\varepsilon + \Delta t^p).$$

Taking the minimum between $E_1$ and $E_2$ one gets the *uniform estimate* [16]

$$\|P^\varepsilon_{\Delta t} - P^\varepsilon\| = O(\Delta t^{p/(r+1)}).$$

[16] F.Golse, S.Jin, D.Levermore '99

# A numerical example

**Broadwell model**

$$\begin{aligned}
&\partial_t \rho + \partial_x m = 0,\\
&\partial_t m + \partial_x z = 0,\\
&\partial_t z + \partial_x m = \frac{1}{\varepsilon}(\rho^2 + m^2 - 2\rho z),
\end{aligned}$$

with $\varepsilon$ is the mean free path. The dynamical variables $\rho$ and $m$ are the density and the momentum respectively, while $z$ represents the flux of momentum.
In the relaxation limit $\varepsilon \to 0$ we obtain

$$\begin{aligned}
&\partial_t \rho + \partial_x m = 0\\
&\partial_t m + \frac{1}{2}\partial_x \left(\rho + \frac{m^2}{\rho}\right) = 0
\end{aligned}$$

(1) Accuracy test for IMEX-RK schemes with smooth initial data and periodic b.c.
(2) Shock test for IMEX-RK schemes.

# Space discretizations

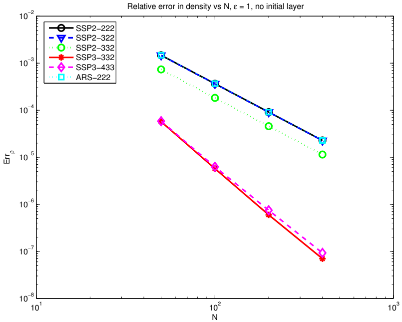
- We can adopt any finite difference/volume or spectral method to approximate the *spatial derivatives*, and use the standard (linear) stability analysis.
- In presence of *shocks and discontinuities* this stability analysis is not sufficient (nonlinear problems can develop discontinuous solutions in finite time even starting from a smooth solution).
- Build spatial discretizations which capture the shock structure and that satisfy some nonlinear stability properties. These methods include *total variation diminishing (TVD)* schemes and *essentially non-oscillatory (ENO)* or *weighted ENO (WENO)* schemes[17].

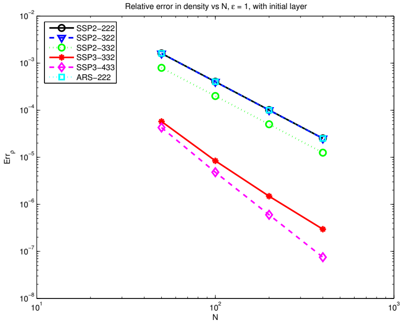
[17] A. Harten '87, T.Chan, X-D.Liu, S.Osher '94, G-S.Jang, C-W.Shu '95

# Convergence rates

| $\varepsilon$ | $1.0$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ | $10^{-6}$ |
|---|---|---|---|---|---|---|---|
| Scheme | Convergence rates for $\rho$ | | | | | | |
| IMEX-ARS | 2.018 | 1.513 | 1.159 | 1.165 | 1.165 | 1.165 | 1.165 |
| IMEX-SSP2 | 2.042 | 2.054 | 2.051 | 2.053 | 2.043 | 2.042 | 2.042 |
| IMEX-ARSF | 2.044 | 2.074 | 2.007 | 1.982 | 2.042 | 2.040 | 2.040 |
| IMEX-SSP2F | 2.050 | 2.064 | 2.061 | 2.065 | 2.056 | 2.055 | 2.055 |
| IMEX-ARS3 | 2.963 | 3.013 | 2.982 | 2.860 | 2.482 | 2.060 | 2.044 |
| IMEX-BHR | 3.119 | 2.994 | 2.930 | 3.117 | 3.146 | 3.211 | 3.187 |
| | Convergence rates for $z$ | | | | | | |
| IMEX-ARS | 1.950 | 1.438 | 1.114 | 1.121 | 1.121 | 1.121 | 1.121 |
| IMEX-SSP2 | 2.027 | 2.045 | 1.965 | 1.501 | 1.309 | 1.302 | 1.302 |
| IMEX-ARSF | 2.031 | 2.174 | 1.762 | 1.596 | 2.061 | 2.040 | 2.039 |
| IMEX-SSP2F | 2.036 | 2.034 | 2.038 | 2.368 | 2.127 | 2.052 | 2.051 |
| IMEX-ARS3 | 2.982 | 2.970 | 2.471 | 2.386 | 2.041 | 2.003 | 1.999 |
| IMEX-BHR | 3.050 | 2.921 | 2.780 | 3.539 | 3.200 | 3.019 | 3.016 |

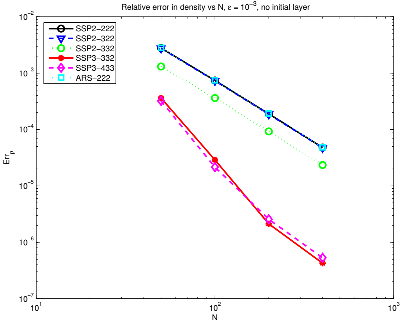
# Convergence rates

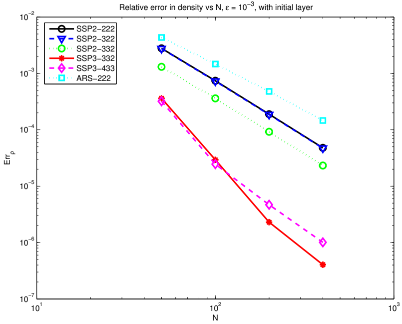



Relative error for different second and third order IMEX-RK schemes for the Broadwell equations with $\epsilon = 1$. Left: no initial layer. Right: initial layer.

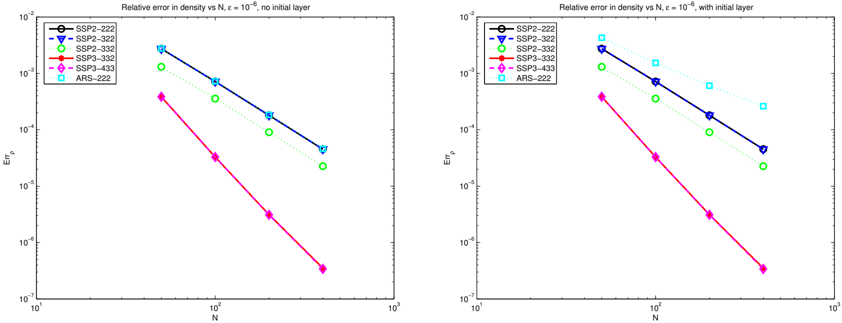
# Convergence rates



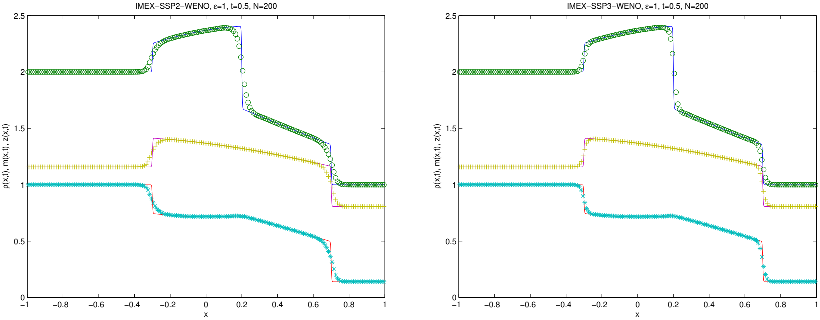

Relative error for different second and third order IMEX-RK schemes for the Broadwell equations with $\epsilon = 10^{-3}$. Left: no initial layer. Right: initial layer.

# Convergence rates



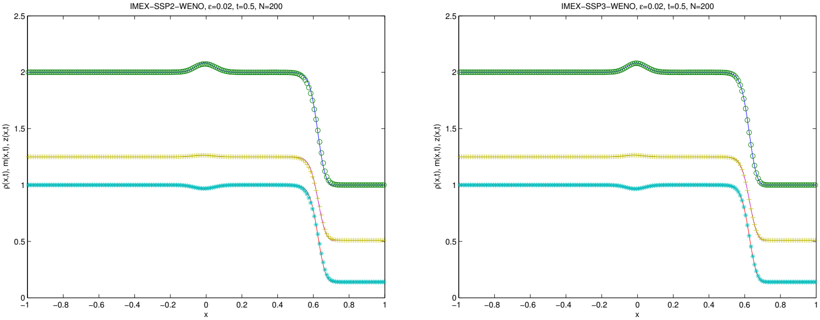

Relative error for different second and third order IMEX-RK schemes for the Broadwell equations with $\epsilon = 10^{-6}$. Left: no initial layer. Right: initial layer.

# Shock test



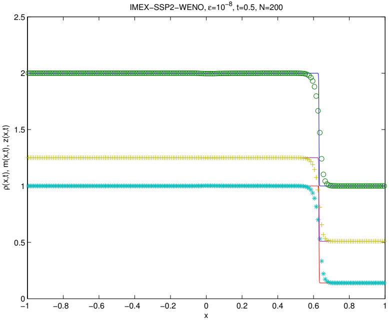

Numerical solution for second and third order SSP IMEX-RK schemes for the Broadwell equations with $\epsilon = 1$

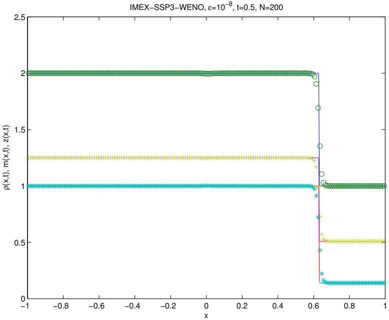
# Shock test




Numerical solution for second and third order SSP IMEX-RK schemes for the Broadwell equations with $\epsilon = 10^{-3}$

# Shock test




Numerical solution for second and third order SSP IMEX-RK schemes for the Broadwell equations with $\epsilon = 10^{-6}$

# Concluding remarks

- *IMEX-schemes* represent a powerful tool for the discretization of multiscale partial differential equations, for example where convection and stiff sources/diffusion are present.
- Other than the AP property, an *efficient implicit solver* is also one of the main ingredients in an IMEX scheme.
- They represent an alternative/complementary approach to domain-decomposition methods. The basic principles can be applied to any PDE where there is the presence of *multiple time/space-scales*.

- Main problem
  - How can we extend the previous approaches to the challenging case of the full Boltzmann equation, where the inversion of the stiff collision operator is computationally prohibitive?