Forword

The solutions of some problems are exponentially asymptotical ($\sim e^{\pm t}$), while the solutions of some problems are algebraically asymptotical ($\sim t^{\pm \gamma}$).

Does there exist that the solutions of some problems are logarithmically asymptotical ($\sim (\log t)^{\pm \gamma}$)?

The answer is positive!

Hadamard-type fractional differential equations!
Forword

From the physical phenomena observed and references available, Hadamard fractional calculus is suitable for describing logarithmic asymptotics, e.g., Lomnitz logarithmic creep law of viscoelastic materials [Lomnitz, 1956], ultra slow process [Denisov & Kantz, 2010], life evolution of Populus euphratica, etc.
Discrete formulae for Caputo-Hadamard fractional derivatives and their applications in large time integration

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Caputo-Hadamard fractional derivative

Definition

Hadamard fractional integral of a given function $f(t)$ with order $\alpha > 0$ is defined by

$$\mathcal{H}D_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left( \log \frac{t}{\tau} \right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}, \; t > a > 0. \quad (1.1)$$

The condition $f(t) \in L^1(a, b)$ is presumed. Omitting does not mean no.
Caputo-Hadamard fractional derivative

**Definition**

**Hadamard fractional derivative** of a given function \( f(t) \) with order \( \alpha (n - 1 < \alpha < n \in \mathbb{Z}^+) \) is defined by

\[
H D^\alpha_{a,t} f(t) = \delta^n \left[ H D^{-(n-\alpha)}_{a,t} f(t) \right] = \frac{1}{\Gamma(n-\alpha)} \delta^n \int_a^t \left( \log \frac{t}{\tau} \right)^{n-\alpha-1} f(\tau) \frac{d\tau}{\tau}, \quad t > a > 0,
\]

where \( \delta = t \frac{d}{dt}, \, \delta^n = \delta (\delta^{n-1}), \, \delta^0 = I. \)

The condition \( f(t) \in AC^n_\delta [a, b] \) is presumed. Omitting does not mean no.
Caputo-Hadamard fractional derivative

**Definition**

Caputo-Hadamard fractional derivative of a given function \( f(t) \) with order \( \alpha \) \((n - 1 < \alpha < n \in \mathbb{Z}^+)\) and \( t > a > 0 \) is defined by

\[
CHD_{a,t}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left( \log \frac{t}{s} \right)^{n-\alpha-1} \delta^n f(s) \frac{ds}{s}, \quad (1.3)
\]

where \( \delta^n f(s) = (s \frac{d}{ds})^n f(s) = \delta(\delta^{n-1} f(s)), \delta^0 f(s) = f(s) \).

The condition \( f(t) \in AC^n_\delta [a, b] \) is presumed. Omitting does not mean no. These two kinds of derivatives have following relation,

\[
CHD_{a,t}^\alpha f(t) = HD_{a,t}^\alpha \left[ f(t) - \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!} \left( \log \frac{t}{a} \right)^k \right],
\]

provided that \( \delta^k f(a) = \delta^k f(t)|_{t=a} \) exist for \( k = 0, n-1 \).
In 2020, L, Li and Wang got an **analytical solution** to a certain linear fractional partial differential equation with the Caputo-Hadamard derivative by a **modified Laplace transform**.

In 2021, L and Li discussed **stability and logarithmic decay** of the solution of Hadamard-type fractional ordinary differential equation.

In 2021, L and Li studied **the blow-up and global existence** of solution to Caputo-Hadamard fractional evolution equation with fractional Laplacian.
However, there are few researches on discrete approximation of the Caputo-Hadamard derivative, except that Gohar, L and Li, 2020, and L, Li and Wang, 2020 derived several numerical approximation formulas of the Caputo-Hadamard derivative with $\alpha \in (0, 1)$.

This report comes from Fan, L and Li, 2022.
Two types of subdivision

The partition of the interval \([a, T]\):

\[ a = t_0 < t_1 < \cdots < t_N = T. \]

**Case A :** Uniform partition

\[ t_k = t_0 + k\tau, \]

\[ \tau = t_k - t_{k-1} = \frac{T - a}{N} (1 \leq k \leq N). \] (1.4)

**Case B :** Special non-uniform partition (uniform partition in the logarithmic sense)

\[ t_k = \exp (\log t_0 + k\tilde{\tau}), \text{ (different nodes)} \]

\[ \tilde{\tau} = \log t_k - \log t_{k-1} = \frac{\log T - \log a}{N} (1 \leq k \leq N). \] (1.5)
Two types of subdivision

For convenience, we define

$$f^k = f(t_k)$$

for the function $f(t)$ on $[a, T]$ and introduce the following operator

$$\nabla_{\log,t} f^{k-\frac{1}{2}} = \frac{f^k - f^{k-1}}{\log \frac{t_k}{t_{k-1}}}.$$
The partition of the interval \([a, T]\):

\[a = t_0 < t_1 < \cdots < t_N = T.\]

**Case A: Uniform partition**

\[t_k = t_0 + k\tau,\]
\[\tau = t_k - t_{k-1} = \frac{T - a}{N} \quad (1 \leq k \leq N).\] (2.1)
We denote the linear interpolation function of \( f(t) \) as \( L_{\log,1,j} f(t) \) on \([t_{j-1}, t_j]\) \((1 \leq j \leq N)\) by \((t_{j-1}, f(t_{j-1})), (t_j, f(t_j))\), that is,

\[
L_{\log,1,j} f(t) = \frac{\log \frac{t}{t_j}}{\log \frac{t_{j-1}}{t_j}} f^{j-1} + \frac{\log \frac{t}{t_{j-1}}}{\log \frac{t_j}{t_{j-1}}} f^j,
\]

(2.2)

and the truncation error on \([t_{j-1}, t_j]\) is

\[
r_1^j(t) = f(t) - L_{\log,1,j} f(t) = \frac{1}{2} \delta^2 f(\eta_j) \log \frac{t}{t_{j-1}} \log \frac{t}{t_j},
\]

(2.3)

where \( \eta_j \in (t_{j-1}, t_j) \).
L1-2 formula with order $0 < \alpha < 1$

We obtain quadratic interpolation function $L_{\log,2,j}f(t)$ on $[t_{j-1}, t_j]$ $(2 \leq j \leq N)$ using $(t_{j-2}, f(t_{j-2}))$, $(t_{j-1}, f(t_{j-1}))$, $(t_j, f(t_j))$,

$$L_{\log,2,j}f(t) = \frac{\log \frac{t}{t_{j-1}} \log \frac{t}{t_j}}{\log \frac{t_{j-2}}{t_{j-1}} \log \frac{t_{j-2}}{t_j}} f^{j-2} + \frac{\log \frac{t}{t_{j-2}} \log \frac{t}{t_j}}{\log \frac{t_{j-1}}{t_{j-2}} \log \frac{t_{j-1}}{t_j}} f^{j-1} + \frac{\log \frac{t}{t_{j-2}} \log \frac{t}{t_j}}{\log \frac{t_j}{t_{j-2}} \log \frac{t_j}{t_{j-1}}} f^j$$

$$= L_{\log,1,j}f(t) + \frac{\nabla_{\log,t} f^{j-\frac{1}{2}} - \nabla_{\log,t} f^{j-\frac{3}{2}}}{\log \frac{t_j}{t_{j-2}}} \log \frac{t}{t_{j-1}} \log \frac{t}{t_j}. \quad (2.4)$$

The truncation error on $[t_{j-1}, t_j]$ is as follows

$$r_2^j(t) = f(t) - L_{\log,2,j}f(t)$$

$$= \frac{1}{6} \delta^3 f(\xi_j) \log \frac{t}{t_{j-2}} \log \frac{t}{t_{j-1}} \log \frac{t}{t_j}, \quad \xi_j \in (t_{j-2}, t_j). \quad (2.5)$$
L1-2 formula with order $0 < \alpha < 1$

From (2.2)-(2.5), we can arrive at

\[
\delta (L_{\log,2,j}f(t)) = \delta (L_{\log,1,j}f(t)) + \frac{\nabla_{\log,t}f_{\frac{j-1}{2}} - \nabla_{\log,t}f_{\frac{j-3}{2}}}{\log \frac{t_j}{t_{j-2}}} \log \frac{t^2}{t_j t_{j-1}},
\]

(2.6)

\[
\delta (L_{\log,1,j}f(t)) = \nabla_{\log,t}f_{\frac{j-1}{2}}.
\]
L1-2 formula with order $0 < \alpha < 1$

\[
CHD_{a,t}^\alpha f(t) \bigg|_{t=t_k} = \frac{1}{\Gamma(1-\alpha)} \left\{ \int_{t_0}^{t_1} \left( \log \frac{t_k}{s} \right)^{-\alpha} \delta (L_{\log,1,1} f(s)) \frac{ds}{s} \right. \\
+ \sum_{j=2}^{k} \int_{t_{j-1}}^{t_j} \left( \log \frac{t_k}{s} \right)^{-\alpha} \delta (L_{\log,2,j} f(s)) \frac{ds}{s} \right\} \\
+ \frac{1}{\Gamma(1-\alpha)} \left\{ \int_{t_0}^{t_1} \left( \log \frac{t_k}{s} \right)^{-\alpha} \delta (r_1^1(s)) \frac{ds}{s} \right. \\
+ \sum_{j=2}^{k} \int_{t_{j-1}}^{t_j} \left( \log \frac{t_k}{s} \right)^{-\alpha} \delta (r_2^j(s)) \frac{ds}{s} \right\} \\
= CHD_{a,t}^\alpha f^k + R^k.
\]
L1-2 formula with order $0 < \alpha < 1$

So L1-2 formula of Caputo-Hadamard fractional derivative with $\alpha \in (0, 1)$ is obtained as follows:

$$C H D_{a,t}^\alpha f^k$$

$$= C H D_{a,t}^\alpha f^k - \frac{1}{\Gamma(2 - \alpha)} \sum_{j=2}^{k} b_{j,k}^{(\alpha)} \left( \nabla_{\log,t} f_{j}^{\frac{1}{2}} - \nabla_{\log,t} f_{j}^{\frac{3}{2}} \right)$$

$$= \frac{1}{\Gamma(2 - \alpha)} \sum_{j=1}^{k} c_{j,k}^{(\alpha)} \left( f_{j}^{\alpha} - f_{j}^{\alpha - 1} \right),$$

(2.8)

where

$$C H D_{a,t}^\alpha f^k = \frac{1}{\Gamma(2 - \alpha)} \sum_{j=1}^{k} a_{j,k}^{(\alpha)} \nabla_{\log,t} f_{j}^{\alpha} \left( f_{j}^{\frac{1}{2}} \right)$$

(L1 formula),
L1-2 formula with order $0 < \alpha < 1$

and

$$c_{j,k}^{(\alpha)} = \begin{cases} \frac{1}{\log \frac{t_1}{t_0}} (a_{1,k}^{(\alpha)} + b_{2,k}^{(\alpha)}), & j = 1, \\ \frac{1}{\log \frac{t_j}{t_{j-1}}} (a_{j,k}^{(\alpha)} - b_{j,k}^{(\alpha)} + b_{j+1,k}^{(\alpha)}), & 2 \leq j \leq k - 1, \\ \frac{1}{\log \frac{t_k}{t_{k-1}}} (a_{k,k}^{(\alpha)} - b_{k,k}^{(\alpha)}), & j = k, \end{cases}$$ (2.9)

$$a_{j,k}^{(\alpha)} = \left( \log \frac{t_k}{t_{j-1}} \right)^{1-\alpha} \left( \log \frac{t_k}{t_j} \right)^{1-\alpha},$$

$$b_{j,k}^{(\alpha)} = \left\{ \log \frac{t_j}{t_{j-1}} \right\} \left[ \left( \log \frac{t_k}{t_j} \right)^{1-\alpha} + \left( \log \frac{t_k}{t_{j-1}} \right)^{1-\alpha} \right]$$

$$+ \frac{2}{2-\alpha} \left[ \left( \log \frac{t_k}{t_j} \right)^{2-\alpha} - \left( \log \frac{t_k}{t_{j-1}} \right)^{2-\alpha} \right] \frac{1}{\log \frac{t_j}{t_{j-2}}}.$$
Theorem

Assuming \( f(t) \in C^3 [a, T] \) and \( 0 < \alpha < 1 \), for uniform partition of the interval \([a, T]\) with \( \tau = t_k - t_{k-1} \), the truncation errors \( R_k \) (\( 1 \leq k \leq N \)) in (2.7) satisfy

\[
|R^1| \leq \frac{\alpha}{2\Gamma(3 - \alpha)} \max_{t_0 \leq t \leq t_1} |\delta^2 f(t)| \left( \log \frac{t_1}{t_0} \right)^{2-\alpha}, \quad k = 1,
\]

\[
|R^k| \leq \frac{\alpha}{8\Gamma(1 - \alpha)} \max_{t_0 \leq t \leq t_1} |\delta^2 f(t)| \left( \log \frac{t_k}{t_1} \right)^{-1-\alpha} \left( \log \frac{t_1}{t_0} \right)^3
\]

\[
+ \frac{1}{12\Gamma(1 - \alpha)} \max_{t_0 \leq t \leq t_{k-1}} |\delta^3 f(t)| \max_{1 \leq l \leq k-1} \left( \log \frac{t_l}{t_{l-1}} \right)^3 \left( \log \frac{t_k}{t_{k-1}} \right)^{-\alpha}
\]

\[
+ \frac{\alpha(5 - \alpha)}{6\Gamma(4 - \alpha)} \max_{t_{k-2} \leq t \leq t_k} |\delta^3 f(t)| \max_{k-1 \leq l \leq k} \left( \log \frac{t_l}{t_{l-1}} \right)^{3-\alpha}, \quad k \geq 2.
\]

(2.10)
For $\alpha \in (0, 1)$, coefficients $b_{j,k}^{(\alpha)}$ ($2 \leq j \leq k$, $2 \leq k \leq N$) in (2.9) with $t_j = t_0 + j\tau$ ($0 \leq j \leq k$) are negative.

The inequalities with $t_j = t_0 + j\tau$ ($1 \leq j \leq k - 2$, $3 \leq k \leq N$) hold

\[
\log \frac{t_{j+2}}{t_{j+1}} \log \frac{t_j}{t_{j-1}} - \left( \log \frac{t_{j+1}}{t_j} \right)^2 > 0. \quad (2.11)
\]
Lemma

For any \( \alpha \in (0, 1) \) and \( t_j = t_0 + j\tau \) (0 \( \leq j \leq k \)), the inequalities with \( a^{(\alpha)}_{j,k} \) and \( b^{(\alpha)}_{j,k} \) in (2.9) hold

\[
a^{(\alpha)}_{j,k} + b^{(\alpha)}_{j+1,k} > 0, \quad 1 \leq j \leq k - 2, \quad 3 \leq k \leq N. \tag{2.12}
\]

Theorem

For any \( \alpha \in (0, 1) \) and \( t_j = t_0 + j\tau \) (0 \( \leq j \leq k \)), coefficients \( c^{(\alpha)}_{j,k} \) (1 \( \leq j \leq k \), 1 \( \leq k \leq N \)) in (2.9) satisfy

\[
c^{(\alpha)}_{j,k} > 0, \quad j \neq k - 1. \tag{2.13}
\]
Remark

For \( j = k - 1, a = 1, T = 2 \) and \( N = 25 \), we find that the sign of \( c_{k-1,k}^{(\alpha)} (3 \leq k \leq N) \) with \( \alpha = 0.1, 0.685, 0.686, 0.9 \) can change.

Figure: The values of \( c_{k-1,k}^{(\alpha)} \) with the uniform partition.
L1-2 formula with order $0 < \alpha < 1$

**Theorem**

For order $\alpha \in (0, 1)$ and sufficiently small step $\tau$, the coefficients $c_{j,k}^{(\alpha)} (1 \leq j \leq k, 2 \leq k \leq N)$ in (2.9) with $t_j = t_0 + j\tau (0 \leq j \leq k)$ satisfy

1. $c_{k,k}^{(\alpha)} > |c_{k-1,k}^{(\alpha)}| (k \geq 2)$,
2. $c_{k,k}^{(\alpha)} > c_{k-2,k}^{(\alpha)} (k \geq 3)$,
3. $c_{k-2,k}^{(\alpha)} > c_{k-3,k}^{(\alpha)} > \cdots > c_{1,k}^{(\alpha)} (k \geq 4)$.  

Discrete formulae for Caputo-Hadamard fractional derivatives and
L2-1_σ formula with 0 < α < 1

We denote the quadratic interpolation function \( \Pi_{\log,2,j}f(t) \) of \( f(t) \) in the sense of logarithm on \([t_{j-1}, t_j]\) (1 ≤ j ≤ k, 1 ≤ k ≤ N − 1) by using the points \((t_{j-1}, f(t_{j-1})), (t_j, f(t_j)), (t_{j+1}, f(t_{j+1}))\),

\[
\Pi_{\log,2,j}f(t) = \frac{\log \frac{t}{t_j} \log \frac{t}{t_{j+1}}}{\log \frac{t_{j-1}}{t_j} \log \frac{t_{j-1}}{t_{j+1}}} f_{j-1} + \frac{\log \frac{t}{t_{j-1}} \log \frac{t}{t_j}}{\log \frac{t_j}{t_{j-1}} \log \frac{t_j}{t_{j+1}}} f_j \\
+ \frac{\log \frac{t}{t_{j-1}} \log \frac{t}{t_{j+1}}}{\log \frac{t_{j+1}}{t_{j-1}} \log \frac{t_{j+1}}{t_j}} f_{j+1},
\]

(2.14)

and the truncation error on \([t_{j-1}, t_j]\),

\[
r_2^j(t) = f(t) - \Pi_{\log,2,j}f(t) \\
= \frac{1}{6} \delta^3 f(\xi_j) \log \frac{t}{t_{j-1}} \log \frac{t}{t_j} \log \frac{t}{t_{j+1}},
\]

(2.15)

where \( \xi_j \in (t_{j-1}, t_{j+1}) \).
Let $\sigma = 1 - \frac{\alpha}{2}$ be a fixed constant and $t_{k+\sigma} = t_k + \sigma \tau$. Then we take $\Pi_{\log,1,k+1} f(t)$ as the linear interpolation function of $f(t)$ on the interval $[t_k, t_{k+\sigma}]$ ($k = 0, 1, \cdots, N - 1$) in the logarithmic sense, using the points $(t_k, f(t_k))$, $(t_{k+1}, f(t_{k+1}))$ to get

$$
\Pi_{\log,1,k+1} f(t) = \frac{\log \frac{t}{t_{k+1}}}{\log \frac{t_k}{t_{k+1}}} f^k + \frac{\log \frac{t}{t_k}}{\log \frac{t_{k+1}}{t_k}} f^{k+1},
$$

(2.16)

and the truncation error on $[t_k, t_{k+\sigma}]$,

$$
r_1^{k+1}(t) = f(t) - \Pi_{\log,1,k+1} f(t) = \frac{1}{2} \delta^2 f(\eta_{k+1}) \log \frac{t}{t_k} \log \frac{t}{t_{k+1}}, \quad \eta_{k+1} \in (t_k, t_{k+1}).
$$

(2.17)
Thus, we can arrive at

\[
\delta \left( \Pi_{\log,2,j} f(t) \right) = \nabla_{\log,t} f^j - \frac{1}{2} + \frac{\nabla_{\log,t} f^j + \frac{1}{2} - \nabla_{\log,t} f^j - \frac{1}{2}}{\log \frac{t_{j+1}}{t_{j-1}}} \log \frac{t^2}{t_j t_{j-1}},
\]

\(2.18\)

\[
\delta \left( \Pi_{\log,1,k+1} f(t) \right) = \nabla_{\log,t} f^k + \frac{1}{2}.
\]
L2-1\(\sigma\) formula with \(0 < \alpha < 1\)

\[
CHD_{a,t}^\alpha f(t) \bigg|_{t = t_{k+\sigma}}
= \frac{1}{\Gamma(1 - \alpha)} \left\{ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left( \log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \delta \left( \Pi_{\log,2,j} f(s) \right) \frac{ds}{s} 
+ \int_{t_k}^{t_{k+\sigma}} \left( \log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \delta \left( \Pi_{\log,1,k+1} f(s) \right) \frac{ds}{s} \right\} 
+ \frac{1}{\Gamma(1 - \alpha)} \left\{ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left( \log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \delta \left( r^j_2(s) \right) \frac{ds}{s} 
+ \int_{t_k}^{t_{k+\sigma}} \left( \log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \delta \left( r^{k+1}_1(s) \right) \frac{ds}{s} \right\}
= CHD_{a,t}^\alpha f^{k+\sigma} + R^{k+\sigma}.
\]
By means of (2.18), we can obtain \(L_2-1_\sigma\) formula with \(0 < \alpha < 1\)

\[
L_2-1_\sigma \text{ formula with } 0 < \alpha < 1
\]

\[
\begin{align*}
C H \mathcal{D}_{a,t}^{\alpha} f^{k+\sigma} &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \left\{ \nabla_{\log,t} f^{j-\frac{1}{2}} \int_{t_{j-1}}^{t_j} \left( \log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \frac{ds}{s} \\
& \quad + \frac{\nabla_{\log,t} f^{j+\frac{1}{2}} - \nabla_{\log,t} f^{j-\frac{1}{2}}}{\log \frac{t_{j+1}}{t_{j-1}}} \int_{t_{j-1}}^{t_j} \left( \log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \log \frac{s^2 \cdot ds}{t_{j-1} t_j} \right\} \\
& \quad + \frac{\nabla_{\log,t} f^{k+\frac{1}{2}}}{\Gamma(1-\alpha)} \int_{t_k}^{t_{k+\sigma}} \left( \log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \frac{ds}{s} \\
& \quad = \frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^{k+1} c_j^{(\alpha,\sigma)} (f^j - f^{j-1}),
\end{align*}
\]

(2.20)
where

\[ c_{j,k}^{(\alpha,\sigma)} = \begin{cases} \frac{1}{\log \frac{t_1}{t_0}} \left( a_{1,k}^{(\alpha,\sigma)} - b_{1,k}^{(\alpha,\sigma)} \right), & j = 1, \\ \frac{1}{\log \frac{t_j}{t_{j-1}}} \left( a_{j,k}^{(\alpha,\sigma)} + b_{j-1,k}^{(\alpha,\sigma)} - b_{j,k}^{(\alpha,\sigma)} \right), & 2 \leq j \leq k, \\ \frac{1}{\log \frac{t_{k+1}}{t_k}} \left( b_{k,k}^{(\alpha,\sigma)} + \left( \log \frac{t_{k+\sigma}}{t_k} \right)^{1-\alpha} \right), & j = k + 1, \end{cases} \]

\[ a_{j,k}^{(\alpha,\sigma)} = \left( \log \frac{t_{k+\sigma}}{t_{j-1}} \right)^{1-\alpha} - \left( \log \frac{t_{k+\sigma}}{t_j} \right)^{1-\alpha}, \]

\[ b_{j,k}^{(\alpha,\sigma)} = \left\{ \frac{2}{2 - \alpha} \left[ \left( \log \frac{t_{k+\sigma}}{t_{j-1}} \right)^{2-\alpha} - \left( \log \frac{t_{k+\sigma}}{t_j} \right)^{2-\alpha} \right] \right. \\ \left. - \log \frac{t_j}{t_{j-1}} \left[ \left( \log \frac{t_{k+\sigma}}{t_j} \right)^{1-\alpha} + \left( \log \frac{t_{k+\sigma}}{t_{j-1}} \right)^{1-\alpha} \right] \right\} \frac{1}{\log \frac{t_{j+1}}{t_{j-1}}}. \]

(2.21)
Theorem

Letting $f(t) \in C^3[a, T]$ and $\alpha \in (0, 1)$, for the fixed $\sigma = 1 - \frac{\alpha}{2}$ and sufficiently small $\tau = \frac{T-a}{N}$, the truncation errors $R^{k+\sigma}$ $(0 \leq k \leq N - 1)$ in (2.19) with $t_k = t_0 + k\tau$ and $t_k + \sigma = t_k + \sigma \tau$ satisfy

$$
|R^{k+\sigma}| \leq \frac{1}{12\Gamma(1-\alpha)} \max_{t_0 \leq t \leq t_{k+1}} |\delta^3 f(t)| \max_{1 \leq l \leq k+1} (\log \frac{t_l}{t_{l-1}})^3 (\log \frac{t_{k+\sigma}}{t_k})^{-\alpha}
$$

$$
+ \left\{ \frac{1}{\Gamma(3-\alpha)} \left( 1 + \frac{\sigma(1-\sigma)}{2} \right) \max_{t_k \leq t \leq t_{k+1}} |\delta^2 f(t)| \right. 
$$

$$
+ \frac{1}{6\Gamma(2-\alpha)} \max_{t_k \leq t \leq t_{k+1}} |\delta^3 f(t)| \right\} (\log \frac{t_{k+1}}{t_k})^2 (\log \frac{t_{k+\sigma}}{t_k})^{1-\alpha}.
$$

(2.22)
L2-1_σ formula with $0 < \alpha < 1$

Lemma

For order $\alpha \in (0, 1)$ and $t_j = t_0 + j \tau$ ($0 \leq j \leq k + 1$), coefficients $b_{j,k}^{(\alpha,\sigma)}$ defined in (2.21) satisfy

$$b_{j,k}^{(\alpha,\sigma)} > 0, \ 1 \leq j \leq k, \ 1 \leq k \leq N - 1.$$  \hspace{1cm} (2.23)

Lemma

For $\alpha \in (0, 1)$ and $t_j = t_0 + j \tau$ ($0 \leq j \leq k + 1$), the inequalities with $a_{j,k}^{(\alpha,\sigma)}$ and $b_{j,k}^{(\alpha,\sigma)}$ in (2.21) hold

$$a_{j,k}^{(\alpha,\sigma)} - b_{j,k}^{(\alpha,\sigma)} > 0, \ 1 \leq j \leq k, \ 1 \leq k \leq N - 1.$$  \hspace{1cm} (2.24)
L2-1_σ formula with 0 < \alpha < 1

**Theorem**

For any order \( \alpha \in (0, 1) \) and \( t_j = t_0 + j\tau \) \((0 \leq j \leq k + 1)\), coefficients \( c_{j,k}^{(\alpha,\sigma)} \) defined in (2.21) satisfy

\[
c_{j,k}^{(\alpha,\sigma)} > 0, \ 1 \leq j \leq k + 1, \ 0 \leq k \leq N - 1. \quad (2.25)
\]

**Theorem**

For order \( \alpha \in (0, 1) \) and sufficiently small \( \tau \), coefficients \( c_{j,k}^{(\alpha,\sigma)} \) \((1 \leq k \leq N - 1)\) defined in (2.21) satisfy

\[
c_{k+1,k}^{(\alpha,\sigma)} > c_{k,k}^{(\alpha,\sigma)} > c_{k-1,k}^{(\alpha,\sigma)} > \cdots > c_{2,k}^{(\alpha,\sigma)} > c_{1,k}^{(\alpha,\sigma)}. \quad (2.26)
\]
H2N2 formula with $1 < \alpha < 2$

Let $t_{k-\frac{1}{2}} = \frac{t_{k-1} + t_k}{2}$, i.e., the arithmetic mean of $t_{k-1}$ and $t_k$. We show the quadratic Hermite interpolation $H_{\log, 2, 0} f(t)$ of $f(t)$ on the interval $[t_0, t_{\frac{1}{2}}]$ in the sense of logarithm using the three points $(t_0, f(t_0)), (t_1, f(t_1)), (t_0, \delta f(t_0))$,.

\[
H_{\log, 2, 0} f(t) = f(t_0) + \delta f(t_0) \log \frac{t}{t_0} + \nabla_{\log, t} f^\frac{1}{2} - \delta f(t_0) \left( \log \frac{t}{t_0} \right)^2,
\]

(2.27)

and the truncation error on $[t_0, t_{\frac{1}{2}}]$,

\[
R_H(t) = f(t) - H_{\log, 2, 0} f(t) = \frac{1}{6} \delta^3 f(\xi_0) \left( \log \frac{t}{t_0} \right)^2 \log \frac{t}{t_1}, \quad \xi_0 \in (t_0, t_1).
\]

(2.28)
Similarly, on the interval \([t_{j-\frac{1}{2}}, t_{j+\frac{1}{2}}]\) \((j = 1, 2, \cdots, N - 1)\), we obtain quadratic Newton interpolation \(N_{\log, 2,j} f(t)\) of the function \(f(t)\) in the logarithmic sense, by means of the points \((t_{j-1}, f(t_{j-1})), (t_j, f(t_j)), (t_{j+1}, f(t_{j+1}))\),

\[
N_{\log, 2,j} f(t) = f(t_{j-1}) + \nabla_{\log,t} f^{j-\frac{1}{2}} \log \frac{t}{t_{j-1}} + \frac{\nabla_{\log,t} f^{j+\frac{1}{2}} - \nabla_{\log,t} f^{j-\frac{1}{2}}}{\log \frac{t_{j+1}}{t_{j-1}}} \log \frac{t}{t_{j-1}} \log \frac{t}{t_j},
\]

and the truncation error on \([t_{j-\frac{1}{2}}, t_{j+\frac{1}{2}}]\) \((1 \leq j \leq N)\),

\[
R^j_N(t) = f(t) - N_{\log, 2,j} f(t)
= \frac{1}{6} \delta^3 f(\xi_j) \log \frac{t}{t_{j-1}} \log \frac{t}{t_j} \log \frac{t}{t_{j+1}}, \xi_j \in (t_{j-1}, t_{j+1}).
\]
Therefore, we have

\[
\begin{align*}
\delta^2 (H_{\log,2,0} f(t)) &= \frac{2(\nabla_{\log,t} f^{\frac{1}{2}} - \delta f(t_0))}{\log \frac{t_1}{t_0}}, \\
\delta^2 (N_{\log,2,j} f(t)) &= \frac{2(\nabla_{\log,t} f^{j+\frac{1}{2}} - \nabla_{\log,t} f^{j-\frac{1}{2}})}{\log \frac{t_{j+1}}{t_{j-1}}}. \\
\end{align*}
\] (2.31)
H2N2 formula with $1 < \alpha < 2$

\[
CHD_{a,t}^\alpha f(t) \bigg|_{t=t_k-\frac{1}{2}} \\
= \left. \frac{1}{\Gamma(2 - \alpha)} \left\{ \int_{t_0}^{t_{\frac{1}{2}}} \left( \log \frac{t_{k-\frac{1}{2}}}{s} \right)^{1-\alpha} \delta^2 H_{\log,2,0} f(s) \frac{ds}{s} \right. \right. \\
+ \left. \sum_{j=1}^{k-1} \int_{t_{j-\frac{1}{2}}}^{t_{\frac{1}{2}}} \left( \log \frac{t_{k-\frac{1}{2}}}{s} \right)^{1-\alpha} \delta^2 N_{\log,2,j} f(s) \frac{ds}{s} \right\} \right. \\
+ \left. \frac{1}{\Gamma(2 - \alpha)} \left\{ \int_{t_0}^{t_{\frac{1}{2}}} \left( \log \frac{t_{k-\frac{1}{2}}}{s} \right)^{1-\alpha} \delta^2 R_H(s) \frac{ds}{s} \right. \right. \\
+ \left. \sum_{j=1}^{k-1} \int_{t_{j-\frac{1}{2}}}^{t_{\frac{1}{2}}} \left( \log \frac{t_{k-\frac{1}{2}}}{s} \right)^{1-\alpha} \delta^2 R_N^j(s) \frac{ds}{s} \right\} \right. \\
= CHD_{a,t}^\alpha f^{k-\frac{1}{2}} + R^{k-\frac{1}{2}}.
\]
By formula (2.31), we can arrive at H2N2 formula with $\alpha \in (1, 2)$

$$\begin{align*}
C_H^\alpha D_{a,t}^k f^{k-\frac{1}{2}} &= \frac{1}{\Gamma(2 - \alpha)} \frac{2(\nabla_{\log,t} f_{\frac{1}{2}} - \delta f(t_0))}{\log \frac{t_1}{t_0}} \int_{t_0}^{t_{\frac{1}{2}}} \left( \log \frac{t_{\frac{1}{2}}}{s} \right)^{1-\alpha} \frac{ds}{s} \\
&+ \frac{1}{\Gamma(2 - \alpha)} \sum_{j=1}^{k-1} \frac{2(\nabla_{\log,t} f_{j+\frac{1}{2}} - \nabla_{\log,t} f_{j-\frac{1}{2}})}{\log \frac{t_{j+1}}{t_{j-1}}} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \left( \log \frac{t_{\frac{1}{2}}}{s} \right)^{1-\alpha} \frac{ds}{s} \\
&= \frac{2}{\Gamma(3 - \alpha)} \sum_{j=1}^{k} c_j^{(\alpha)} (f^j - f^{j-1}) - \frac{2}{\Gamma(3 - \alpha)} a_0^{(\alpha)} \delta f(t_0),
\end{align*}$$

(2.33)
H2N2 formula with $1 < \alpha < 2$

where

\[
\begin{align*}
    c_{j,k}^{(\alpha)} &= \begin{cases} 
        \frac{1}{\log \frac{t_1}{t_0}} (a_{0,k}^{(\alpha)} - a_{1,k}^{(\alpha)}), & j = 1, \\
        \frac{1}{\log \frac{t_j}{t_{j-1}}} (a_{j-1,k}^{(\alpha)} - a_{j,k}^{(\alpha)}), & 2 \leq j \leq k - 1, \\
        \frac{1}{\log \frac{t_k}{t_{k-1}}} a_{k-1,k}^{(\alpha)}, & j = k, \\
        \frac{1}{\log \frac{t_1}{t_0}} (\log \frac{t_k - \frac{1}{2}}{t_0})^{2-\alpha} - (\log \frac{t_k - \frac{1}{2}}{\frac{1}{2}})^{2-\alpha}, & j = 0, \\
        \frac{1}{\log \frac{t_j}{t_{j-1}}} (\log \frac{t_k - \frac{1}{2}}{t_{j-\frac{1}{2}}})^{2-\alpha} - (\log \frac{t_k - \frac{1}{2}}{t_{j+\frac{1}{2}}})^{2-\alpha}, & 1 \leq j \leq k - 1.
    \end{cases}
\end{align*}
\]

(2.34) Discrete formulae for Caputo-Hadamard fractional derivatives and their applications in large time integration.
H2N2 formula with $1 < \alpha < 2$

**Theorem**

Let $f(t) \in C^3[a, T]$ and $1 < \alpha < 2$, for the sufficiently small $\tau$, the truncation errors $R^{k-\frac{1}{2}}$ ($1 \leq k \leq N$) in (2.32) hold

\[
|R^{k-\frac{1}{2}}| \leq \frac{1}{\Gamma(3 - \alpha)} \max_{t_0 \leq t \leq t_1} |\delta^3 f(t)| \log \frac{t_1}{t_0} \left( \log \frac{t_1}{t_0} \right)^{2-\alpha}, \quad k = 1,
\]

\[
|R^{k-\frac{1}{2}}| \leq \frac{1}{\Gamma(2 - \alpha)} \max_{t_0 \leq t \leq t_{k-1}} |\delta^3 f(t)| \left\{ \frac{17}{6} \max_{1 \leq l \leq k-1} \left( \log \frac{t_l}{t_{l-1}} \right)^2 \left( \log \frac{t_{k-\frac{1}{2}}}{t_{k-3/2}} \right)^{1-\alpha} \right. \\
+ \left. \frac{3}{8} \log \frac{T}{a} \max_{1 \leq l \leq k-1} \left( \log \frac{t_l}{t_{l-1}} \right)^3 \left( \log \frac{t_{k-\frac{1}{2}}}{t_{k-3/2}} \right)^{-\alpha} \right\}
\]

\[
+ \frac{1}{\Gamma(3 - \alpha)} \max_{t_{k-2} \leq t \leq t_k} |\delta^3 f(t)| \max_{1 \leq l \leq k-1} \log \frac{t_l}{t_{l-1}} \left( \log \frac{t_{k-\frac{1}{2}}}{t_{k-3/2}} \right)^{2-\alpha}, \quad k \geq 2.
\]

(2.35)
H2N2 formula with $1 < \alpha < 2$

**Theorem**

For $\alpha \in (1, 2)$, coefficients $c_{j,k}^{(\alpha)}$ in (2.34) with $t_j = t_0 + j\tau$ and $t_{j-\frac{1}{2}} = t_{j-1} + \frac{1}{2}\tau (1 \leq j \leq k, 1 \leq k \leq N)$ hold

$$c_{k,k}^{(\alpha)} > 0, \quad c_{j,k}^{(\alpha)} < 0 (2 \leq j \leq k - 1).$$

(2.36)

**Theorem**

For $\alpha \in (1, 2)$ and sufficiently small $\tau$, coefficients $c_{j,k}^{(\alpha)}$ in (2.34) with $t_j = t_0 + j\tau$ and $t_{j-\frac{1}{2}} = t_{j-1} + \frac{1}{2}\tau (1 \leq j \leq k, 1 \leq k \leq N)$ hold

1. $c_{1,k}^{(\alpha)} > c_{2,k}^{(\alpha)} > c_{3,k}^{(\alpha)} > \cdots > c_{k-1,k}^{(\alpha)}$;
2. $c_{k,k}^{(\alpha)} > \left| c_{k-1,k}^{(\alpha)} \right| (k \geq 2)$. 

Discrete formulae for Caputo-Hadamard fractional derivatives and
The partition of the interval \([a, T]\):

\[ a = t_0 < t_1 < \cdots < t_N = T. \]

**Case B**: Special non-uniform partition (uniform partition in the logarithmic sense)

\[ t_k = \exp (\log t_0 + k \tilde{\tau}) , \quad \text{(different nodes)} \]

\[ \tilde{\tau} = \log t_k - \log t_{k-1} = \frac{\log T - \log a}{N} \quad (1 \leq k \leq N) . \]  

(3.1)
The **L1-2 formula** can be rewritten as the following form under uniform division in the logarithmic sense (Case B).

\[
\text{CH} \mathcal{D}_a^\alpha f^k = \frac{(\bar{\tau})^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{j=1}^{k} \tilde{c}_{j,k}^{(\alpha)} (f^j - f^{j-1}), \tag{3.2}
\]

where

\[
\begin{align*}
\tilde{c}_{j,k}^{(\alpha)} &= \begin{cases} 
\tilde{a}_{1,k}^{(\alpha)} + \tilde{b}_{2,k}^{(\alpha)}, & j = 1, \\
\tilde{a}_{j,k}^{(\alpha)} - \tilde{b}_{j,k}^{(\alpha)} + \tilde{b}_{j+1,k}^{(\alpha)}, & 2 \leq j \leq k - 1, \\
\tilde{a}_{k,k}^{(\alpha)} - \tilde{b}_{k,k}^{(\alpha)}, & j = k,
\end{cases} \\
\tilde{a}_{j,k}^{(\alpha)} &= (k - j + 1)^{1-\alpha} - (k - j)^{1-\alpha}, \\
\tilde{b}_{j,k}^{(\alpha)} &= \frac{1}{2} \left[ (k - j)^{1-\alpha} + (k - j + 1)^{1-\alpha} \right] \\
&\quad + \frac{1}{2 - \alpha} \left[ (k - j)^{2-\alpha} - (k - j + 1)^{2-\alpha} \right].
\end{align*}
\]
**Theorem**

*Letting* $f(t) \in C^3[\alpha, T]$ *and* $0 < \alpha < 1$, *for* $t_k = \exp \left( \log t_0 + k \tilde{\tau} \right)$ *and* $\tilde{\tau} = \log t_k - \log t_{k-1} (0 \leq j \leq k)$, *then the truncation errors* $R^k (1 \leq k \leq N)$ *in* (2.7) *satisfy*

\[
|R^1| \leq \frac{\alpha}{2 \Gamma(3 - \alpha)} \max_{t_0 \leq t \leq t_1} |\delta^2 f(t)| \tilde{\tau}^{2-\alpha},
\]

\[
|R^k| \leq \frac{\alpha}{8 \Gamma(1 - \alpha)} \max_{t_0 \leq t \leq t_1} |\delta^2 f(t)| \left( \log \frac{t_k}{t_1} \right)^{-1-\alpha} \tilde{\tau}^3
\]

\[
+ \max_{t_0 \leq t \leq t_k} |\delta^3 f(t)| \left[ \frac{1}{12 \Gamma(1 - \alpha)} + \frac{\alpha(5 - \alpha)}{6 \Gamma(4 - \alpha)} \right] \tilde{\tau}^{3-\alpha}, \quad k \geq 2.
\]

(3.4)
The coefficients obtained in this case are the same as those obtained by L1-2 formula of the Caputo derivative.

Lemma

For $\alpha \in (0, 1)$, coefficients $\tilde{c}_{j,k}^{(\alpha)}$ ($1 \leq j \leq k$, $1 \leq k \leq N$) satisfy

1) $k = 1$: $\tilde{c}_{1,1}^{(\alpha)} = 1$,

2) $k = 2$:
   1) $\tilde{c}_{1,2}^{(\alpha)} = 2^{1-\alpha} - \left(\frac{1}{2} + \frac{1}{2-\alpha}\right) \in \left(-\frac{1}{2}, 1\right)$,
   2) $|\tilde{c}_{1,2}^{(\alpha)}| < \tilde{c}_{2,2}^{(\alpha)}$,

3) $k \geq 3$:
   1) $\tilde{c}_{k,k}^{(\alpha)} > |\tilde{c}_{k-1,k}^{(\alpha)}|$,  
   2) $\tilde{c}_{k,k}^{(\alpha)} > \tilde{c}_{k-2,k}^{(\alpha)}$,
   3) $\tilde{c}_{j,k}^{(\alpha)} > 0$, $j \neq k - 1$,
   4) $\tilde{c}_{k-2,k}^{(\alpha)} > \tilde{c}_{k-3,k}^{(\alpha)} > \cdots > \tilde{c}_{1,k}^{(\alpha)}$,
   5) $\sum_{j=1}^{k} \tilde{c}_{j,k}^{(\alpha)} = k^{1-\alpha}$. 

Discrete formulae for Caputo-Hadamard fractional derivatives and
L2-1\(\sigma\) formula with 0 < \(\alpha\) < 1

For \(\sigma = 1 - \frac{\alpha}{2}\) and \(t_{k+\sigma} = \exp(\log t_k + \sigma \tilde{\tau})\), the L2-1\(\sigma\) formula on uniform partition in the logarithmic sense (Case B) can be

\[
CH\mathcal{D}_a^{(\alpha)} f^{k+\sigma} = \frac{(\tilde{\tau})^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k+1} \tilde{c}_{j,k}^{(\alpha,\sigma)} (f^j - f^{j-1}),
\]

where

\[
\tilde{c}_{j,k}^{(\alpha,\sigma)} = \begin{cases} 
\tilde{a}_{1,k}^{(\alpha,\sigma)} - \tilde{b}_{1,k}^{(\alpha,\sigma)}, & j = 1, \\
\tilde{a}_{j,k}^{(\alpha,\sigma)} + \tilde{b}_{j-1,k}^{(\alpha,\sigma)} - \tilde{b}_{j,k}^{(\alpha,\sigma)}, & 2 \leq j \leq k, \\
\tilde{b}_{k,k}^{(\alpha,\sigma)} + \sigma^{1-\alpha}, & j = k + 1,
\end{cases}
\]

\[
\tilde{a}_{j,k}^{(\alpha,\sigma)} = (k + \sigma - j + 1)^{1-\alpha} - (k + \sigma - j)^{1-\alpha},
\]

\[
\tilde{b}_{j,k}^{(\alpha,\sigma)} = \frac{1}{2-\alpha} \left[ (k + \sigma - j + 1)^{2-\alpha} - (k + \sigma - j)^{2-\alpha} \right] - \frac{1}{2} \left[ (k + \sigma - j + 1)^{1-\alpha} + (k + \sigma - j)^{1-\alpha} \right].
\]
Letting \( f(t) \in C^3[a, T] \) and \( \alpha \in (0, 1) \), for the fixed \( \sigma = 1 - \frac{\alpha}{2} \), the truncation errors \( R^{k+\sigma} \) \((0 \leq k \leq N - 1)\) defined in (2.19) with \( t_k = \exp (\log t_0 + k\tilde{\tau}) \) and \( t_{k+\sigma} = \exp (\log t_0 + (k + \sigma)\tilde{\tau}) \) satisfy

\[
|R^{k+\sigma}| \leq \left\{ \frac{\sigma^{-\alpha}}{12\Gamma(1-\alpha)} \max_{t_0 \leq t \leq t_{k+1}} |\delta^3 f(t)| + \frac{\sigma^{1-\alpha}}{6\Gamma(2-\alpha)} \max_{t_k \leq t \leq t_{k+1}} |\delta^3 f(t)| \right\} \tilde{\tau}^{3-\alpha}.
\]

Discrete formulae for Caputo-Hadamard fractional derivatives and
L2-1_σ formula with 0 < α < 1

The coefficients \( \tilde{c}^{(\alpha,\sigma)}_{j,k} \) in (3.6) are identical to the coefficients of L2-1_σ formula of the Caputo derivative which satisfy the following properties.

**Lemma**

**For any order** \( \alpha \in (0, 1), \sigma = 1 - \frac{\alpha}{2} \) **and** \( t_j = \exp (\log t_0 + j \tilde{\tau}) \)

(0 ≤ j ≤ k + 1), coefficients \( \tilde{c}^{(\alpha,\sigma)}_{j,k} \) (1 ≤ j ≤ k + 1) in (3.6) satisfy

1. \( \tilde{c}^{(\alpha,\sigma)}_{j,k} > \frac{1-\alpha}{2} (k - j + 1 + \sigma)^{-\alpha} \),

2. \( \tilde{c}^{(\alpha,\sigma)}_{k+1,k} > \tilde{c}^{(\alpha,\sigma)}_{k,k} > \tilde{c}^{(\alpha,\sigma)}_{k-1,k} > \cdots > \tilde{c}^{(\alpha,\sigma)}_{2,k} > \tilde{c}^{(\alpha,\sigma)}_{1,k} \),

3. \( (2\sigma - 1) \tilde{c}^{(\alpha,\sigma)}_{k+1,k} > \sigma \tilde{c}^{(\alpha,\sigma)}_{k,k} \).
H2N2 formula with \( 1 < \alpha < 2 \)

Let \( t_{k-\frac{1}{2}} = \exp \left( \log t_k - \frac{1}{2} \tilde{\tau} \right) = \sqrt{t_{k-1}t_k} \) (geometric mean). The H2N2 formula on uniform partition in the logarithmic sense (Case B) can be

\[
C H D_{\alpha,t}^{\alpha} f^{k-\frac{1}{2}} = \frac{2(\tilde{\tau})^{-\alpha}}{\Gamma(3 - \alpha)} \sum_{j=1}^{k} \tilde{c}_{j,k}^{(\alpha)} (f^j - f^{j-1}) - \frac{2(\tilde{\tau})^{1-\alpha}}{\Gamma(3 - \alpha)} \tilde{a}_{0,k}^{(\alpha)} \delta f(t_0),
\]

(3.8)

where

\[
\tilde{c}_{j,k}^{(\alpha)} = \begin{cases} 
\tilde{a}_{0,k}^{(\alpha)} - \tilde{a}_{1,k}^{(\alpha)}, & j = 1, \\
\tilde{a}_{j-1,k}^{(\alpha)} - \tilde{a}_{j,k}^{(\alpha)}, & 2 \leq j \leq k - 1, \\
\tilde{a}_{k-1,k}^{(\alpha)}, & j = k,
\end{cases}
\]

(3.9)

\[
\tilde{a}_{j,k}^{(\alpha)} = \begin{cases} 
(k - \frac{1}{2})^{2-\alpha} - (k - 1)^{2-\alpha}, & j = 0, \\
\frac{1}{2}[(k - j)^{2-\alpha} - (k - j - 1)^{2-\alpha}], & 1 \leq j \leq k - 1.
\end{cases}
\]
H2N2 formula with \(1 < \alpha < 2\)

Theorem

Supposing \(f(t) \in C^3[a, T]\) and \(\alpha \in (1, 2)\), the following inequalities for the truncation errors \(R^{k-\frac{1}{2}}\) \((1 \leq k \leq N)\) defined in (2.32) with \(t_j = \exp(\log t_0 + j\tilde{\tau})\) and \(\bar{t}_{j-\frac{1}{2}} = \exp(\log t_0 + (j - \frac{1}{2})\tilde{\tau})\) hold

\[
|R^{k-\frac{1}{2}}| \leq \frac{1}{2^{2-\alpha}\Gamma(3 - \alpha)} \max_{t_0 \leq t \leq t_1} |\delta^3 f(t)|^{\frac{3-\alpha}{\tilde{\tau}}}, \quad k = 1,
\]

\[
|R^{k-\frac{1}{2}}| \leq \max_{t_0 \leq t \leq t_k} |\delta^3 f(t)| \left\{ \frac{5}{3\Gamma(2 - \alpha)} + \frac{1}{\Gamma(3 - \alpha)} \right\}^{\frac{3-\alpha}{\tilde{\tau}}}, \quad k \geq 2.
\]

(3.10)
The coefficients \( \tilde{c}_{j,k}^{(\alpha)} \) are similar to the coefficients of the H2N2 formula of the Caputo derivative.

**Lemma**

For coefficients \( \tilde{c}_{j,k}^{(\alpha)} \) defined in (3.9) with \( t_j = \exp (\log t_0 + j\tilde{\tau}) \) and \( \tilde{t}_{j - \frac{1}{2}} = \exp \left( \log t_0 + \left( j - \frac{1}{2} \right)\tilde{\tau} \right) \) (1 \( \leq j \leq k \), 1 \( \leq k \leq N \)) and \( \alpha \in (1, 2) \), it holds that

1. \( \tilde{c}_{k,k}^{(\alpha)} > 0, \quad \tilde{c}_{j,k}^{(\alpha)} < 0 \) (1 \( \leq j \leq k - 1 \)),
2. \( \tilde{c}_{1,k}^{(\alpha)} > \tilde{c}_{2,k}^{(\alpha)} > \cdots > \tilde{c}_{k-1,k}^{(\alpha)} \) (\( k \geq 3 \)),
3. \( \left| \tilde{c}_{k-1,k}^{(\alpha)} \right| < \tilde{c}_{k,k}^{(\alpha)} \).
Consider the following fractional ordinary differential equation with initial value condition and $\alpha \in (0, 1)$

\[
\begin{cases}
C_H D_{a,t}^\alpha u(t) = g(t), & t \in [a, T], \\
u(a) = u_a.
\end{cases}
\] (4.1)

Let $a = 1$, $T = 2$, $u_a = 0$ and $g(t) = \frac{6}{\Gamma(4-\alpha)} (\log t)^{3-\alpha}$, and then the exact solution $u(t) = (\log t)^3$. 
Table: Errors and convergence results for L1-2 formula.

<table>
<thead>
<tr>
<th>Case</th>
<th>N</th>
<th>α</th>
<th>Error 0.1</th>
<th>Rate 0.1</th>
<th>Error 0.5</th>
<th>Rate 0.5</th>
<th>Error 0.9</th>
<th>Rate 0.9</th>
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</table>
### Numerical examples

**Table:** Errors and convergence results for L2-1$_\alpha$ formula.

<table>
<thead>
<tr>
<th>Case</th>
<th>N</th>
<th>$\alpha$ 0.1 Error</th>
<th>Rate</th>
<th>$\alpha$ 0.5 Error</th>
<th>Rate</th>
<th>$\alpha$ 0.9 Error</th>
<th>Rate</th>
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</table>

Discrete formulae for Caputo-Hadamard fractional derivatives and
Example

For $\alpha \in (1, 2)$, we consider the fractional initial value problem

\[
\begin{cases}
C H D_{a,t}^{\alpha} u(t) = g(t), & t \in [a, T], \\
u(a) = u_a, \quad \delta u(a) = v_a.
\end{cases}
\] (4.2)

Let $a = 1$, $T = 2$, $u_a = v_a = 0$ and $g(t) = \frac{6}{\Gamma(4-\alpha)} (\log t)^{3-\alpha}$, so
we can derive the exact solution $u(t) = (\log t)^3$. 

Numerical examples
## Numerical examples

**Table:** Errors and convergence results for H2N2 formula

<table>
<thead>
<tr>
<th>Case</th>
<th>α</th>
<th>1.2 Error</th>
<th>Rate</th>
<th>1.5 Error</th>
<th>Rate</th>
<th>1.8 Error</th>
<th>Rate</th>
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</thead>
<tbody>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>–</td>
<td>8.8392E-04</td>
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<td>1.4671</td>
<td>7.7736E-04</td>
<td>1.1887</td>
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</tbody>
</table>

Discrete formulae for Caputo-Hadamard fractional derivatives and applications in large time integration.
Consider the following Lorenz system with Caputo-Hadamard fractional derivative with \( \alpha \in (0, 1) \)

\[
\begin{align*}
CHD_{a,t}^\alpha x_1(t) &= \bar{a}(x_2(t) - x_1(t)), \\
CHD_{a,t}^\alpha x_2(t) &= \bar{c}x_1(t) - x_2(t) - x_1(t)x_3(t), \\
CHD_{a,t}^\alpha x_3(t) &= x_1(t)x_2(t) - \bar{b}x_3(t),
\end{align*}
\]

where \( t > a > 0, \bar{a}, \bar{b} \) and \( \bar{c} \) are intrinsic parameters. For the given parameter value \((\bar{a}, \bar{b}, \bar{c}) = (10, \frac{8}{3}, 200)\), we choose the initial value \((x_1(a), x_2(a), x_3(a)) = (x_1(2.5), x_2(2.5), x_3(2.5)) = (5, 3, 9)\), \( t \in [a, T] = [2.5, T], T = 60 \).
### Lorenz system

**Table:** The existence of chaotic attractors with changed $\alpha$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\max {l_1, l_2, l_3}$</th>
<th>Existence of chaotic attractor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000</td>
<td>1.374808979490</td>
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<tr>
<td>0.95000</td>
<td>0.805883701541</td>
<td>yes</td>
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<tr>
<td>0.93750</td>
<td>0.362285530845</td>
<td>yes</td>
</tr>
<tr>
<td><strong>0.93125</strong></td>
<td><strong>2.686635511486</strong></td>
<td><strong>yes</strong></td>
</tr>
<tr>
<td>0.92500</td>
<td>-1.214511641728</td>
<td>no</td>
</tr>
<tr>
<td>0.90000</td>
<td>-1.917961361415</td>
<td>no</td>
</tr>
</tbody>
</table>
Lorenz system

Figure: The chaotic attractor of system (5.1) using L1-2 method.
Lorenz system

Figure: The chaotic attractor of system (5.1) using L1-2 method.
Lorenz system

Figure: The phase portrait of system (5.1) using L1-2 method. No chaotic attractors in this case.
Let us consider the following Bagley-Torvik problem with Caputo-Hadamard fractional derivative of order $\alpha = 3/2$,

$$\begin{cases}
A \frac{d^2 y(t)}{dt^2} + B CHD_{1,t}^{3/2} y(t) + C y(t) = g(t), & t > 1, \\
y(1) = 0, \quad \delta y(t)|_{t=1} = 0,
\end{cases} \quad (5.2)$$

where parameters $A$, $B$ and $C$ are constants. For numerical calculation, choose $T = 100$, $A = 1$, $B = C = 0.5$ and the source term

$$g(t) = \begin{cases}
8, & 1 \leq t \leq 2, \\
0, & (100 = T \geq) t > 2.
\end{cases} \quad (5.3)$$
Figure: The solutions of Bagley-Torvik system by H2N2 formula.
Thank you all for your attention!!!