Numerical approximations to ψ fractional derivative

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Outline

ψ-fractional integral

Definition

Assume $f \in L^1(a,b)$. And let $\psi \in C^1(a,+\infty)$ be a strictly increasing function with $\psi(t) \to +\infty$ as $t \to +\infty$. Then the (left) ψ-fractional integral of f is defined by

$$ {}_{\psi}\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\psi(t) - \psi(s))^{\alpha-1} f(s)\psi'(s)\mathrm{d}s, \ \ t > a. \qquad (1.1) $$

The condition $f \in L^1(a,b)$ is a sufficient condition.

ψ-fractional integral

- The original idea of fractional integral of a function by another one (i.e., a ψ-fractional integral) was proposed in [Liouville, 1835]. This idea was more distinctly formulated thirty years later in [Holmgren, 1865].
- It can be seen as a product of the fusion of Stieltjes integral $\int_\Omega f \mathrm{d}g$ [Claesson, Hörmander，1970] and Riemann-Liouville fractional integral [Samko, Kilbas, Marichev, 1993].
- But in their definitions, the condition

$$\psi(t) \to +\infty \quad \text{as} \quad t \to +\infty \tag{1.2}$$

was not imposed, so the equilibrium to the corresponding fractional differential equation is always stable where the fractional derivative was induced by the original definition of ψ-fractional integral — for details, see [L, Li, 2023].

Chinese version of [1]

ψ-fractional integral

So hereafter, whenever the formula (1.1) is used, the condition (1.2) is always implied. In particular, (1.2) is satisfied in the following cases:

- If $\psi(t) = t$, then it becomes the well-known Riemann-Liouville fractional integral [Samko, Kilbas, Marichev, 1993].
- If $\psi(t) = \log t$ and $a > 0$, then it becomes the Hadamard fractional integral [Hadamard, 1892].
- If $\psi(t) = e^t$, then it becomes the exponential fractional integral [L, Li, 2022].

ψ-Caputo fractional derivative

Let $AC[a,b]$ denote the space of absolutely continuous function on the finite interval $[a,b]$. For $n \in \mathbb{Z}^+$, set $AC^n[a,b] = \{f \in C[a,b] : f^{(n-1)} \in AC[a,b]\}$.

For all suitable functions f and ψ, for $n = 0, 1, \cdots$, define inductively

$$\delta_\psi^n f(s) := \left(\frac{1}{\psi'(s)}\frac{\mathrm{d}}{\mathrm{d}s}\right)^n f(s) = \delta_\psi\left(\delta_\psi^{n-1} f(s)\right), \text{ with } \delta_\psi^0 f(s) := f(s).$$

Set $AC^n_{\delta_\psi}[a,b] = \left\{f : [a,b] \to \mathbb{R} \text{ with } \delta_\psi^{n-1} f \in AC[a,b]\right\}$.

Definition

Let $n-1 < \alpha < n \in \mathbb{Z}^+$. Let $f \in AC^n_{\delta_\psi}[a,b]$. Assume that $\psi \in C^n[a,+\infty)$ is a strictly increasing function with $\psi(t) \to +\infty$ as $t \to +\infty$, and $\psi'(t) \neq 0$ for all t. Then the (left) ψ-Caputo fractional derivative is defined by

$$_{C\psi}\mathrm{D}^\alpha_{a,t} f(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t)-\psi(s))^{n-\alpha-1} \delta_\psi^n f(s) \psi'(s) \mathrm{d}s, \ a < t \leq b.$$

The hypothesis $f \in AC^n_{\delta_\psi}[a,b]$ is sufficient to ensure existence of $_{C\psi}\mathrm{D}^\alpha_{a,t} f(t)$.

ψ-Riemann-Liouville fractional derivative

Definition

Let $n-1 < \alpha < n \in \mathbb{Z}^+$ and let $f \in AC^n_{\delta_\psi}[a,b]$. Assume that $\psi(t) \in C^n[a,+\infty)$ is a strictly increasing function with $\psi(t) \to +\infty$ as $t \to +\infty$, and $\psi'(t) \neq 0$ for all t. Then the (left) ψ-Riemann-Liouville fractional derivative (call ψ fractional derivative for brevity) is defined by

$$ {}_{\psi}\mathrm{D}^{\alpha}_{a,t} f(t) := \frac{1}{\Gamma(n-\alpha)} \delta^n_\psi \left(\int_a^t (\psi(t)-\psi(s))^{n-\alpha-1} f(s)\psi'(s)\mathrm{d}s \right), \quad a < t \leq b. $$

The same hypothesis $f \in AC^n_{\delta_\psi}[a,b]$ is also sufficient to ensure existence of ${}_{\psi}\mathrm{D}^{\alpha}_{a,t} f(t)$.

The relationship between two fractional derivatives

Note that the above definitions of the two fractional derivatives are not equivalent. In fact, the relationship between them is given by the following equality [L, Li, 2022], for $\alpha \in (n-1, n)$ with $n \in \mathbb{Z}^+$:

$$ {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t) = {}_{\psi}\mathrm{D}^{\alpha}_{a,t} f(t) - \sum_{k=0}^{n-1} \frac{\delta_{\psi}^{k} f(a)}{\Gamma(k-\alpha+1)} \left(\psi(t)-\psi(a)\right)^{k-\alpha}, $$

provided that all $\delta_{\psi}^{k} f(a) := \delta_{\psi}^{k} f(t)|_{t=a}$ $(k = 0, \cdots, n-1)$ exist. Since the above relation enables us to convert one form of derivative to the other, in this paper we shall discuss only the ψ-Caputo fractional derivative.

Numerical discretisation

- In the standard case $\psi(t) = t$, the L1 discretisation [Oldham, Spanier, 1974; Sun, Wu, 2006; Lin, Xu, 2007], L1-2 discretisation [Gao, Sun, Zhang, 2014], and L2-1_σ discretisation [Alikhanov, 2015] are familiar approximations of the usual Caputo derivative ${}_C\mathrm{D}^{\alpha}_{a,t}f$ of order $\alpha \in (0,1)$, while the L2 discretisation [Oldham, Spanier, 1974] and H2N2 discretisation [L, Zeng, 2015; Shen, L, Sun, 2020] have been derived for ${}_C\mathrm{D}^{\alpha}_{a,t}f$ of order $\alpha \in (1,2)$.
- When $\psi(t) = \log t$, extensions of these discretisations have been constructed [Gohar, L, Li, 2020; Fan, L, Li, 2022] for the Caputo-Hadamard derivative of order $\alpha \in (0,1) \cup (1,2)$.

Interpolation in the sense of the function $\psi(t)$

Set $f^k = f(t_k)$ and define $\nabla_{\psi,t} f^{k-\frac{1}{2}} = \dfrac{f^k - f^{k-1}}{\psi(t_k) - \psi(t_{k-1})}$. Then we define several useful interpolation formulas.

- Linear Lagrange interpolation in the sense of the function $\psi(t)$:

$$
\begin{aligned}
f(t) &= \left\{ \frac{\psi(t_j) - \psi(t)}{\psi(t_j) - \psi(t_{j-1})} f^{j-1} + \frac{\psi(t) - \psi(t_{j-1})}{\psi(t_j) - \psi(t_{j-1})} f^j \right\} \\
&\quad + \frac{1}{2}\delta_\psi^2 f(\eta_j)(\psi(t) - \psi(t_{j-1}))(\psi(t) - \psi(t_j)) \\
&= L_{\psi,1,j} f(t) + r_1^j(t), \ \ \eta_j \in (t_{j-1}, t_j), \ \ t \in [t_{j-1}, t_j], \ 1 \le j \le N.
\end{aligned}
$$

- Quadratic Lagrange interpolation in the sense of the function $\psi(t)$:

$$
\begin{aligned}
f(t) = &\left\{ \frac{(\psi(t) - \psi(t_j))(\psi(t) - \psi(t_{j+1}))}{(\psi(t_{j-1}) - \psi(t_j))(\psi(t_{j-1}) - \psi(t_{j+1}))} f^{j-1} + \frac{(\psi(t) - \psi(t_{j-1}))}{(\psi(t_j) - \psi(t_{j-1}))} \right. \\
&\left. \times \frac{(\psi(t) - \psi(t_{j+1}))}{(\psi(t_j) - \psi(t_{j+1}))} f^j + \frac{(\psi(t) - \psi(t_{j-1}))(\psi(t) - \psi(t_j))}{(\psi(t_{j+1}) - \psi(t_{j-1}))(\psi(t_{j+1}) - \psi(t_j))} f^{j+1} \right\} \\
&+ \frac{1}{6}\delta_\psi^3 f(\xi_j)(\psi(t) - \psi(t_{j-1}))(\psi(t) - \psi(t_j))(\psi(t) - \psi(t_{j+1})) \\
&= L_{\psi,2,j} f(t) + r_2^j(t), \ \ \xi_j \in (t_{j-1}, t_{j+1}), \ \ t \in [t_{j-1}, t_{j+1}], \ 1 \le j \le N-1.
\end{aligned}
$$

Interpolation in the sense of the function $\psi(t)$

- Quadratic Hermite interpolation in the sense of $\psi(t)$:

$$f(t)=\left\{f(t_0)+\delta_\psi f(t_0)(\psi(t)-\psi(t_0))+\frac{\nabla_{\psi,t}f^{\frac{1}{2}}-\delta_\psi f(t_0)}{\psi(t_1)-\psi(t_0)}\times(\psi(t)-\psi(t_0))^2\right\}+\frac{1}{6}\delta_\psi^3 f(\xi_0)(\psi(t)-\psi(t_0))^2(\psi(t)-\psi(t_1))$$
$$=H_{\psi,2,0}f(t)+R_H(t),\ \xi_0\in(t_0,t_1),\ t\in[t_0,t_1].$$

- Quadratic Newton interpolation in the sense of the function $\psi(t)$:

$$f(t)=\left\{f(t_{j-1})+\nabla_{\psi,t}f^{j-\frac{1}{2}}(\psi(t)-\psi(t_{j-1}))+\frac{\nabla_{\psi,t}f^{j+\frac{1}{2}}-\nabla_{\psi,t}f^{j-\frac{1}{2}}}{\psi(t_{j+1})-\psi(t_{j-1})}(\psi(t)-\psi(t_{j-1}))(\psi(t)-\psi(t_j))\right\}$$
$$+\frac{1}{6}\delta_\psi^3 f(\xi_j)(\psi(t)-\psi(t_{j-1}))(\psi(t)-\psi(t_j))(\psi(t)-\psi(t_{j+1}))$$
$$=N_{\psi,2,j}f(t)+R_N^j(t),\ \xi_j\in(t_{j-1},t_{j+1}),\ t\in[t_{j-1},t_{j+1}],\ 1\le j\le N-1.$$

Partitions

For any given T, the interval $[a, T]$ is partitioned as $a = t_0 < t_1 < \cdots < t_N = T$ where $N \in \mathbb{Z}^+$. Here two types of partitions will be used.

Type A : Uniform partition

$$t_k = t_0 + k\tau, \; \tau = t_k - t_{k-1} = \frac{T - a}{N}.$$

Type B : Special non-uniform partition

$$t_k = \psi^{-1}\big(\psi(t_0) + k\widetilde{\tau}\big), \; i.e., \; \widetilde{\tau} = \psi(t_k) - \psi(t_{k-1}) = \frac{\psi(T) - \psi(a)}{N},$$

where ψ^{-1} denotes the inverse of the function ψ, which exists since ψ is strictly increasing. The image $\{\psi(t_k)\}_{k=1}^{N}$ of this partition is uniform in the interval $[\psi(a), \psi(T)]$.

L1 discretisation for Type A

For $\alpha \in (0,1)$ and $t = t_k\,(1 \le k \le N)$, the linear interpolation in the sense of the function $\psi(t)$ is used to get

$$
\begin{aligned}
{}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t)\Big|_{t=t_k} &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \nabla_{\psi,t} f^{j-\frac{1}{2}} \int_{t_{j-1}}^{t_j} \left(\psi(t_k)-\psi(s)\right)^{-\alpha} \psi'(s)\mathrm{d}s \\
&+ \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left(\psi(t_k)-\psi(s)\right)^{-\alpha} \delta_{\psi}\left(r_1^j(s)\right) \psi'(s)\mathrm{d}s.
\end{aligned}
$$

The L1 discretisation of ψ-Caputo fractional derivative ${}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t)$ with $\alpha \in (0,1)$ at $t = t_k\,(1 \le k \le N)$ on the **Type A** partition is

$$
{}_{C\psi}\mathscr{D}^{\alpha}_{a,t} f^k := \frac{1}{\Gamma(2-\alpha)} \left\{ a^{(\alpha)}_{k,k} f^k - \sum_{j=1}^{k-1} \left(a^{(\alpha)}_{j+1,k} - a^{(\alpha)}_{j,k} \right) f^j - a^{(\alpha)}_{1,k} f^0 \right\},
$$

where

$$
a^{(\alpha)}_{j,k} = \frac{\left(\psi(t_k)-\psi(t_{j-1})\right)^{1-\alpha} - \left(\psi(t_k)-\psi(t_j)\right)^{1-\alpha}}{\psi(t_j)-\psi(t_{j-1})}, \quad j = 1, 2, \cdots, k.
$$

L1 discretisation for **Type A**

Theorem

Let $\alpha \in (0,1)$ and $\delta_\psi^2 f(t) \in C[a,T]$. Then the truncation error satisfies

$$\left|R^k\right| \leq \frac{1}{8\Gamma(1-\alpha)} \max_{t_0 \leq t \leq t_{k-1}} |\delta_\psi^2 f(t)| \max_{1 \leq l \leq k-1} (\psi(t_l) - \psi(t_{l-1}))^2 (\psi(t_k) - \psi(t_{k-1}))^{-\alpha}$$
$$+ \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_{k-1} \leq t \leq t_k} |\delta_\psi^2 f(t)| (\psi(t_k) - \psi(t_{k-1}))^{2-\alpha};$$

that is,

$$\left|R^k\right| \leq C\tau^{2-\alpha},$$

in which

$$C = \frac{1}{8\Gamma(1-\alpha)} \max_{t_0 \leq t \leq t_{k-1}} |\delta_\psi^2 f(t)| \max_{1 \leq l \leq k-1} (\psi'(\xi_l))^2 (\psi'(\xi_k))^{-\alpha}$$
$$+ \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_{k-1} \leq t \leq t_k} |\delta_\psi^2 f(t)| (\psi'(\xi_k))^{2-\alpha},$$

where $\xi_l \in (t_{l-1}, t_l)$ for $l = 1, 2, \cdots, k$.

L1 discretisation for **Type A**

Theorem

For $\alpha \in (0,1)$ *and the* **Type A** *partition, the coefficients* $a_{j,k}^{(\alpha)}$ *in* (13) *satisfy*

$$a_{k,k}^{(\alpha)} > a_{k-1,k}^{(\alpha)} > \cdots > a_{1,k}^{(\alpha)} > 0 \quad \textit{for } 1 \le j \le k \le N.$$

L1-2 discretisation for **Type A**

Let $\alpha \in (0,1)$. Consider a mesh point $t = t_k\,(1 \leq k \leq N)$ from the **Type A** uniform division. Applying the linear and quadratic Lagrange interpolations in the sense of function $\psi(t)$ to discretize the ψ-Caputo derivative ${}_{C\psi}\mathrm{D}^{\alpha}_{a,t}f(t)$, one gets

$$
\begin{aligned}
&{}_{C\psi}\mathrm{D}^{\alpha}_{a,t}f(t)\Big|_{t=t_k} \\
&= \frac{1}{\Gamma(1-\alpha)}\left\{\int_{t_0}^{t_1}\big(\psi(t_k)-\psi(s)\big)^{-\alpha}\delta_\psi\big(L_{\psi,1,1}f(s)+r_1^1(s)\big)\psi'(s)\mathrm{d}s\right. \\
&\quad \left. + \sum_{j=2}^{k}\int_{t_{j-1}}^{t_j}\big(\psi(t_k)-\psi(s)\big)^{-\alpha}\delta_\psi\big(L_{\psi,2,j-1}f(s)+r_2^{j-1}(s)\big)\psi'(s)\mathrm{d}s\right\}.
\end{aligned}
$$

L1-2 discretisation for **Type A**

The L1-2 discretisation of ${}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t)\Big|_{t=t_k}$ on the **Type A** is

$$ {}_{C\psi}\mathbb{D}^{\alpha}_{a,t} f^k := \frac{1}{\Gamma(2-\alpha)} \left\{ c^{(\alpha)}_{k,k} f^k - \sum_{j=1}^{k-1} \left(c^{(\alpha)}_{j+1,k} - c^{(\alpha)}_{j,k} \right) f^j - c^{(\alpha)}_{1,k} f^0 \right\}, $$

where

$$ c^{(\alpha)}_{j,k} = \begin{cases} \dfrac{1}{\psi(t_1)-\psi(t_0)} \left(a^{(\alpha)}_{1,k} - b^{(\alpha)}_{2,k} \right), \; j=1, \\ \dfrac{1}{\psi(t_j)-\psi(t_{j-1})} \left(a^{(\alpha)}_{j,k} + b^{(\alpha)}_{j,k} - b^{(\alpha)}_{j+1,k} \right), \; 2 \le j \le k-1, \\ \dfrac{1}{\psi(t_k)-\psi(t_{k-1})} \left(a^{(\alpha)}_{k,k} + b^{(\alpha)}_{k,k} \right), \; j=k, \end{cases} $$

$$ a^{(\alpha)}_{j,k} = \left(\psi(t_k)-\psi(t_{j-1})\right)^{1-\alpha} - \left(\psi(t_k)-\psi(t_j)\right)^{1-\alpha}, $$

$$ b^{(\alpha)}_{j,k} = \frac{1}{\psi(t_j)-\psi(t_{j-2})} \left\{ \frac{2}{2-\alpha} \left[\left(\psi(t_k)-\psi(t_{j-1})\right)^{2-\alpha} - \left(\psi(t_k)-\psi(t_j)\right)^{2-\alpha} \right] \right. $$
$$ \left. - \left(\psi(t_j)-\psi(t_{j-1})\right) \left[\left(\psi(t_k)-\psi(t_{j-1})\right)^{1-\alpha} + \left(\psi(t_k)-\psi(t_j)\right)^{1-\alpha} \right] \right\}. $$

L1-2 discretisation for Type A

Theorem

Let $\delta_\psi^3 f(t) \in C[a,T]$ and $\alpha \in (0,1)$. Then for $1 \le k \le N$, the truncation error R^k of the L1-2 discretisation on the uniform partition satisfies

$$\left|R^1\right| \le \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_\psi^2 f(t)\right| \left(\psi(t_1)-\psi(t_0)\right)^{2-\alpha}, \quad k=1,$$

$$\left|R^k\right| \le \frac{\alpha}{8\Gamma(1-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_\psi^2 f(t)\right| \left(\psi(t_k)-\psi(t_1)\right)^{-1-\alpha} \left(\psi(t_1)-\psi(t_0)\right)^3$$

$$+ \frac{1}{12\Gamma(1-\alpha)} \max_{t_0 \le t \le t_{k-1}} |\delta_\psi^3 f(t)| \max_{1 \le l \le k-1} \left(\psi(t_l)-\psi(t_{l-1})\right)^3 \left(\psi(t_k)-\psi(t_{k-1})\right)^{-\alpha}$$

$$+ \frac{\alpha}{3\Gamma(2-\alpha)} \max_{t_{k-2} \le t \le t_k} \left|\delta_\psi^3 f(t)\right| \max_{k-1 \le l \le k} \left(\psi(t_l)-\psi(t_{l-1})\right)^{3-\alpha}, \quad k \ge 2;$$

L1-2 discretisation for **Type A**

Theorem

that is,

$$\left|R^1\right| \leq C_1\tau^{2-\alpha}, \;\; k=1; \;\; \left|R^k\right| \leq C_2\tau^{3-\alpha}, \;\; k\geq 2,$$

in which

$$\begin{aligned}
C_1 &= \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_0\leq t\leq t_1}\left|\delta_\psi^2 f(t)\right| \left(\psi'(\xi_1)\right)^{2-\alpha}, \\
C_2 &= \frac{\alpha}{8\Gamma(1-\alpha)} \max_{t_0\leq t\leq t_1}\left|\delta_\psi^2 f(t)\right| \left(\psi(t_k)-\psi(t_1)\right)^{-1-\alpha}\left(\psi'(\xi_1)\right)^3\tau^\alpha \\
&\quad + \frac{1}{12\Gamma(1-\alpha)} \max_{t_0\leq t\leq t_{k-1}}|\delta_\psi^3 f(t)| \max_{1\leq l\leq k-1}\left(\psi'(\xi_l)\right)^3\left(\psi'(\xi_k)\right)^{-\alpha} \\
&\quad + \frac{\alpha}{3\Gamma(2-\alpha)} \max_{t_{k-2}\leq t\leq t_k}\left|\delta_\psi^3 f(t)\right| \max_{k-1\leq l\leq k}\left(\psi'(\xi_l)\right)^{3-\alpha},
\end{aligned}$$

where $\xi_l \in (t_{l-1}, t_l)$, $l = 1, 2, \cdots, k$.

L1-2 discretisation for **Type A**

Theorem

Let $\alpha \in (0,1)$. Let the mesh width $\tau = (T-a)/N$ be sufficiently small. Assume that ψ'' exists in $[a,T]$. Then for $1 \le j \le k \le N$, the coefficients $c_{j,k}^{(\alpha)}$ satisfy

$$c_{j,k}^{(\alpha)} > 0,\ j \neq k-1,$$

but the sign of $c_{k-1,k}^{(\alpha)}$ is uncertain for $k \ge 2$.

Theorem

Let $\alpha \in (0,1)$. Let the mesh width $\tau = t_k - t_{k-1} = \dfrac{T-a}{N}$ be sufficiently small. Assume that $\psi \in C^2[a,T]$ and $\psi' \ge M > 0$ for a constant M. Then for $1 \le j \le k \le N$, the coefficients $c_{j,k}^{(\alpha)}$ satisfy

(1) $c_{1,k}^{(\alpha)} < c_{2,k}^{(\alpha)} < \cdots < c_{k-2,k}^{(\alpha)},\ k \ge 4,$ (2) $\left|c_{k-1,k}^{(\alpha)}\right| < c_{k,k}^{(\alpha)},\ k \ge 2,$

(3) $c_{k-2,k}^{(\alpha)} < c_{k,k}^{(\alpha)},\ k \ge 3.$

L1-2 discretisation for Type A

Table: The coefficients $c_{j,k}^{(\alpha)}$ of L1-2 discretisation

$k=1$	$c_{1,1}^{(\alpha)}$					
$k=2$	$\boxed{c_{1,2}^{(\alpha)}}$	$c_{2,2}^{(\alpha)}$				
$k=3$	$c_{1,3}^{(\alpha)}$	$\boxed{c_{2,3}^{(\alpha)}}$	$c_{3,3}^{(\alpha)}$			
$k=4$	$c_{1,4}^{(\alpha)}$	$c_{2,4}^{(\alpha)}$	$\boxed{c_{3,4}^{(\alpha)}}$	$c_{4,4}^{(\alpha)}$		
$k=5$	$c_{1,5}^{(\alpha)}$	$c_{2,5}^{(\alpha)}$	$c_{3,5}^{(\alpha)}$	$\boxed{c_{4,5}^{(\alpha)}}$	$c_{5,5}^{(\alpha)}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

[1] The signs of the coefficients $c_{k-1,k}^{(\alpha)}$ in the boxes are uncertain.

[2] The first $k-2$ coefficients in row k are strictly increasing, and $|c_{k-1,k}^{(\alpha)}| < c_{k,k}^{(\alpha)}$, but the relative sizes of $c_{k-2,k}^{(\alpha)}$ and $|c_{k-1,k}^{(\alpha)}|$ is unknown.

L2-1$_\sigma$ discretisation for Type A

Let $\alpha \in (0,1)$. We shall use the **Type A** uniform partition. For $0 \leq k \leq N-1$, set $t_{k+\sigma} := t_k + \sigma\tau$ where the constant $\sigma := 1 - \dfrac{\alpha}{2}$. Using the linear and quadratic Lagrange interpolants in the sense of $\psi(t)$, we write

$$
\begin{aligned}
& {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t)\Big|_{t=t_{k+\sigma}} \\
&= \frac{1}{\Gamma(1-\alpha)} \Bigg\{ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \big(\psi(t_{k+\sigma}) - \psi(s)\big)^{-\alpha} \delta_\psi \left(L_{\psi,2,j} f(s) + r_2^j(s) \right) \psi'(s)\mathrm{d}s \\
&\quad + \int_{t_k}^{t_{k+\sigma}} \big(\psi(t_{k+\sigma}) - \psi(s)\big)^{-\alpha} \delta_\psi \left(L_{\psi,1,k+1} f(s) + r_1^{k+1}(s) \right) \psi'(s)\mathrm{d}s \Bigg\}.
\end{aligned}
$$

L2-1$_\sigma$ discretisation for Type A

Then the L2-1$_\sigma$ discretisation for ${}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t)$ is

$$ {}_{C\psi}\mathfrak{D}^{\alpha}_{a,t} f^{k+\sigma} := \frac{1}{\Gamma(2-\alpha)} \left\{ c^{(\alpha,\sigma)}_{k+1,k} f^{k+1} - \sum_{j=1}^{k} \left(c^{(\alpha,\sigma)}_{j+1,k} - c^{(\alpha,\sigma)}_{j,k} \right) f^{j} - c^{(\alpha,\sigma)}_{1,k} f^{0} \right\}, $$

where

$$ c^{(\alpha,\sigma)}_{j,k} = \begin{cases} \dfrac{1}{\psi(t_1)-\psi(t_0)} \left(a^{(\alpha,\sigma)}_{1,k} - b^{(\alpha,\sigma)}_{1,k} \right), & j = 1, \\ \dfrac{1}{\psi(t_j)-\psi(t_{j-1})} \left(a^{(\alpha,\sigma)}_{j,k} + b^{(\alpha,\sigma)}_{j-1,k} - b^{(\alpha,\sigma)}_{j,k} \right), & 2 \le j \le k, \\ \dfrac{1}{\psi(t_{k+1})-\psi(t_k)} \left(a^{(\alpha,\sigma)}_{k+1,k} + b^{(\alpha,\sigma)}_{k,k} \right), & j = k+1, \end{cases} $$

$$ a^{(\alpha,\sigma)}_{j,k} = \left(\psi(t_{k+\sigma}) - \psi(t_{j-1})\right)^{1-\alpha} - \left(\psi(t_{k+\sigma}) - \psi(t_j)\right)^{1-\alpha}, \quad 1 \le j \le k, $$

$$ a^{(\alpha,\sigma)}_{k+1,k} = \left(\psi(t_{k+\sigma}) - \psi(t_k)\right)^{1-\alpha}, $$

$$ b^{(\alpha,\sigma)}_{j,k} = \frac{1}{\psi(t_{j+1})-\psi(t_{j-1})} \left\{ \frac{2}{2-\alpha} \left[\left(\psi(t_{k+\sigma}) - \psi(t_{j-1})\right)^{2-\alpha} - \left(\psi(t_{k+\sigma}) - \psi(t_j)\right)^{2-\alpha} \right] \right. $$
$$ \left. - \left(\psi(t_j) - \psi(t_{j-1})\right) \left[\left(\psi(t_{k+\sigma}) - \psi(t_{j-1})\right)^{1-\alpha} + \left(\psi(t_{k+\sigma}) - \psi(t_j)\right)^{1-\alpha} \right] \right\}. $$

L2-1$_\sigma$ discretisation for Type A

Theorem

Let $\alpha \in (0,1)$ and $\delta_\psi^3 f \in C[a,T]$. For τ sufficiently small and $\sigma = 1 - \frac{\alpha}{2}$, the truncation errors $R^{k+\sigma}$ $(0 \le k \le N-1)$ have the following estimates:

$$\begin{aligned}
&\left|R^{k+\sigma}\right| \\
\le & \frac{1}{12\Gamma(1-\alpha)} \max_{t_0 \le t \le t_{k+1}} \left|\delta_\psi^3 f(t)\right| \max_{1 \le l \le k+1} \left(\psi(t_l) - \psi(t_{l-1})\right)^3 \left(\psi(t_{k+\sigma}) - \psi(t_k)\right)^{-\alpha} \\
& + \left\{ \frac{1}{\Gamma(3-\alpha)} \left(\frac{\sigma(1-\sigma)}{2} \frac{|\psi''(t_k)|}{(\psi'(t_k))^2} + 1 \right) \max_{t_k \le t \le t_{k+1}} \left|\delta_\psi^2 f(t)\right| \right. \\
& \left. + \frac{1}{6\Gamma(2-\alpha)} \max_{t_k \le t \le t_{k+1}} \left|\delta_\psi^3 f(t)\right| \right\} \left(\psi(t_{k+1}) - \psi(t_k)\right)^2 \left(\psi(t_{k+\sigma}) - \psi(t_k)\right)^{1-\alpha},
\end{aligned}$$

L2-1$_\sigma$ discretisation for Type A

Theorem

that is,

$$\left|R^{k+\sigma}\right| \leq C\tau^{3-\alpha},$$

where

$$\begin{aligned}C =& \frac{1}{12\Gamma(1-\alpha)} \max_{t_0\leq t\leq t_{k+1}} \left|\delta_\psi^3 f(t)\right| \max_{1\leq l\leq k+1} \left(\psi'(\xi_l)\right)^3 \left(\psi'(\eta_{k+1})\right)^{-\alpha} \\ &+ \left\{\frac{1}{\Gamma(3-\alpha)} \left(\frac{\sigma(1-\sigma)}{2} \frac{|\psi''(t_k)|}{(\psi'(t_k))^2} + 1\right) \max_{t_k\leq t\leq t_{k+1}} \left|\delta_\psi^2 f(t)\right| \right. \\ &\left. + \frac{1}{6\Gamma(2-\alpha)} \max_{t_k\leq t\leq t_{k+1}} \left|\delta_\psi^3 f(t)\right| \right\} \left(\psi'(\xi_{k+1})\right)^2 \left(\psi'(\eta_{k+1})\right)^{1-\alpha},\end{aligned}$$

where $\xi_l \in (t_{l-1}, t_l)$ for $l = 1, 2, \cdots, k+1$ and $\eta_{k+1} \in (t_k, t_{k+\sigma})$.

L2-1_σ discretisation for **Type A**

Theorem

Let $\alpha \in (0,1)$ and $t_j = t_0 + j\tau$ for $0 \le j \le k+1$. The coefficients $c_{j,k}^{(\alpha,\sigma)}$ are positive, that is,

$$c_{j,k}^{(\alpha,\sigma)} > 0 \quad \text{for } 1 \le j \le k+1 \text{ and } 0 \le k \le N-1.$$

Theorem

Let $\alpha \in (0,1)$. Assume that $\psi \in C^2[a,T]$ and $\psi' \ge M > 0$ for a constant M. Then for all sufficiently small $\tau = t_{k+1} - t_k = \dfrac{T-a}{N}$, the coefficients $c_{j,k}^{(\alpha,\sigma)}$ satisfy

$$c_{1,k}^{(\alpha,\sigma)} < c_{2,k}^{(\alpha,\sigma)} < \cdots < c_{k,k}^{(\alpha,\sigma)} < c_{k+1,k}^{(\alpha,\sigma)} \quad \text{for } 0 \le k \le N-1.$$

L2 discretisation for Type A

Consider the ψ-Caputo derivative ${}_{C\psi}\mathrm{D}^{\alpha}_{a,t}f$ with $\alpha \in (1,2)$. The following formula ${}_{C\psi}\mathbf{D}^{\alpha}_{a,t}f^k$ is the L2 discretisation

$$
\begin{aligned}
&{}_{C\psi}\mathrm{D}^{\alpha}_{a,t}f(t)\Big|_{t=t_k} \\
&\approx \frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{k}\frac{2\left(\nabla_{\psi,t}f^{j+\frac{1}{2}}-\nabla_{\psi,t}f^{j-\frac{1}{2}}\right)}{\psi(t_{j+1})-\psi(t_{j-1})}\int_{t_{j-1}}^{t_j}\left(\psi(t_k)-\psi(s)\right)^{1-\alpha}\psi'(s)\mathrm{d}s \\
&= \frac{2}{\Gamma(3-\alpha)}\sum_{j=1}^{k+1}c^{(\alpha)}_{j,k}\left(f^j-f^{j-1}\right) \\
&:= {}_{C\psi}\mathbf{D}^{\alpha}_{a,t}f^k,
\end{aligned}
$$

where

$$
c^{(\alpha)}_{j,k} = \begin{cases}
\dfrac{-1}{\psi(t_1)-\psi(t_0)}a^{(\alpha)}_{1,k}, & j=1, \\
\dfrac{1}{\psi(t_j)-\psi(t_{j-1})}\left(a^{(\alpha)}_{j-1,k}-a^{(\alpha)}_{j,k}\right), & 2\le j\le k, \\
\dfrac{1}{\psi(t_{k+1})-\psi(t_k)}a^{(\alpha)}_{k,k}, & j=k+1,
\end{cases}
$$

$$
a^{(\alpha)}_{j,k} = \frac{1}{\psi(t_{j+1})-\psi(t_{j-1})}\left[\left(\psi(t_k)-\psi(t_{j-1})\right)^{2-\alpha}-\left(\psi(t_k)-\psi(t_j)\right)^{2-\alpha}\right].
$$

L2 discretisation for **Type A**

Theorem

Let $\alpha \in (1,2)$ and $\delta_\psi^3 f \in C[a,T]$. Then the truncation errors $R^k (1 \leq k \leq N-1)$ on the **Type A** partition satisfy

$$\left|R^k\right| \leq \frac{5\left(\psi(T)-\psi(a)\right)^{2-\alpha}}{3\Gamma(3-\alpha)} \max_{t_0 \leq t \leq t_{k+1}} \left|\delta_\psi^3 f(t)\right| \max_{1 \leq l \leq k+1} \left(\psi(t_l)-\psi(t_{l-1})\right),$$

that is,

$$\left|R^k\right| \leq C\tau,$$

where

$$C = \frac{5\left(\psi(T)-\psi(a)\right)^{2-\alpha}}{3\Gamma(3-\alpha)} \max_{t_0 \leq t \leq t_{k+1}} \left|\delta_\psi^3 f(t)\right| \max_{1 \leq l \leq k+1} \psi'(\xi_l), \quad \xi_l \in (t_{l-1}, t_l).$$

L2 discretisation for Type A

Theorem

Let $\alpha \in (1,2)$. Assume that $\psi \in C^1[a,T]$. Then for all sufficiently small $\tau = t_k - t_{k-1} = \dfrac{T-a}{N}$, one has the following conclusions. For $1 \le k \le N-1$, the coefficients $c_{j,k}^{(\alpha)}$ satisfy

$$c_{j,k}^{(\alpha)} < 0 \ \text{ for } 1 \le j \le k, \quad \text{and } c_{k+1,k}^{(\alpha)} > 0.$$

Theorem

Let $\alpha \in (1,2)$. Assume that $\psi \in C^2[a,T]$, $\psi' \ge M > 0$ for a constant M, and ψ''' exists on $[a,T]$. Then for all sufficiently small τ, the coefficients $c_{j,k}^{(\alpha)}$ satisfy

(1) $c_{j+1,k}^{(\alpha)} < c_{j,k}^{(\alpha)}$ for $2 \le j \le k-1$,

(2) $\left|c_{k,k}^{(\alpha)}\right| < c_{k+1,k}^{(\alpha)}$ for $k \ge 2$,

(3) $c_{1,k}^{(\alpha)} < c_{2,k}^{(\alpha)}$ for $k \ge 3$.

L2 discretisation for Type A

Table: The coefficients $c_{j,k}^{(\alpha)}$ of L2 discretisation

$k=1$	$c_{1,1}^{(\alpha)}$	$c_{2,1}^{(\alpha)}$					
$k=2$	$c_{1,2}^{(\alpha)}$	$c_{2,2}^{(\alpha)}$	$c_{3,2}^{(\alpha)}$				
$k=3$	$c_{1,3}^{(\alpha)}$	$c_{2,3}^{(\alpha)}$	$c_{3,3}^{(\alpha)}$	$c_{4,3}^{(\alpha)}$			
$k=4$	$c_{1,4}^{(\alpha)}$	$c_{2,4}^{(\alpha)}$	$c_{3,4}^{(\alpha)}$	$c_{4,4}^{(\alpha)}$	$c_{5,4}^{(\alpha)}$		
$k=5$	$c_{1,5}^{(\alpha)}$	$c_{2,5}^{(\alpha)}$	$c_{3,5}^{(\alpha)}$	$c_{4,5}^{(\alpha)}$	$c_{5,5}^{(\alpha)}$	$c_{6,5}^{(\alpha)}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

1. For $1 \le j \le k$, $c_{j,k}^{(\alpha)} < 0$, and $c_{k+1,k}^{(\alpha)} > 0$.
2. In row k, the 2nd to the k-th coefficients are strictly decreasing and $|c_{k,k}^{(\alpha)}| < c_{k+1,k}^{(\alpha)}$.
3. For $k \ge 3$, $c_{1,k}^{(\alpha)} < c_{2,k}^{(\alpha)}$ and the size between $c_{1,2}^{(\alpha)}$ and $c_{2,2}^{(\alpha)}$ is uncertain.
4. $|c_{1,1}^{(\alpha)}| > (<)\, c_{2,1}^{(\alpha)}$ if $\psi(t)$ is a strictly lower (upper) convex function. $|c_{1,1}^{(\alpha)}| = c_{2,1}^{(\alpha)}$ as $\tau \to 0$.

H2N2 discretisation for Type A

We continue to discuss the case $\alpha \in (1,2)$. Using quadratic Hermite interpolation and quadratic Newton interpolation in the sense of the function $\psi(t)$ on the **Type A** uniform partition with $t_{k-\frac{1}{2}} = \frac{1}{2}\big(t_{k-1} + t_k\big)$ for $1 \le k \le N$, one has

$$
\begin{aligned}
&{}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t)\Big|_{t=t_{k-\frac{1}{2}}} \\
&= \frac{1}{\Gamma(2-\alpha)}\left\{ \int_{t_0}^{t_{\frac{1}{2}}} \Big(\psi(t_{k-\frac{1}{2}}) - \psi(s)\Big)^{1-\alpha} \delta_\psi^2 \big(H_{\psi,2,0} f(s) + R_H(s)\big)\psi'(s)\mathrm{d}s \right. \\
&\quad \left. + \sum_{j=1}^{k-1} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \Big(\psi(t_{k-\frac{1}{2}}) - \psi(s)\Big)^{1-\alpha} \delta_\psi^2 \big(N_{\psi,2,j} f(s) + R_N^j(s)\big)\psi'(s)\mathrm{d}s \right\}.
\end{aligned}
$$

H2N2 discretisation for Type A

Then, we define the H2N2 discretisation of the ψ-Caputo derivative ${}_{C\psi}\mathbb{D}^{\alpha}_{a,t}f(t_{k-\frac{1}{2}})$ for $1 \leq k \leq N$ by

$$ {}_{C\psi}\mathbb{D}^{\alpha}_{a,t}f^{k-\frac{1}{2}} := \frac{2}{\Gamma(3-\alpha)}\sum_{j=1}^{k} c_{j,k}^{(\alpha)}\left(f^{j}-f^{j-1}\right) - \frac{2}{\Gamma(3-\alpha)} a_{0,k}^{(\alpha)}\,\delta_\psi f(t_0), $$

where

$$ c_{j,k}^{(\alpha)} = \begin{cases} \dfrac{1}{\psi(t_j)-\psi(t_{j-1})}\left(a_{j-1,k}^{(\alpha)} - a_{j,k}^{(\alpha)}\right), & 1 \leq j \leq k-1, \\ \dfrac{1}{\psi(t_k)-\psi(t_{k-1})} a_{k-1,k}^{(\alpha)}, & j = k, \end{cases} $$

$$ a_{0,k}^{(\alpha)} = \frac{1}{\psi(t_1)-\psi(t_0)}\left[\left(\psi(t_{k-\frac{1}{2}})-\psi(t_0)\right)^{2-\alpha} - \left(\psi(t_{k-\frac{1}{2}})-\psi(t_{\frac{1}{2}})\right)^{2-\alpha}\right], $$

$$ a_{j,k}^{(\alpha)} = \frac{1}{\psi(t_{j+1})-\psi(t_{j-1})}\left[\left(\psi(t_{k-\frac{1}{2}})-\psi(t_{j-\frac{1}{2}})\right)^{2-\alpha} - \left(\psi(t_{k-\frac{1}{2}})-\psi(t_{j+\frac{1}{2}})\right)^{2-\alpha}\right]. $$

H2N2 discretisation for **Type A**

Theorem

Let $\alpha \in (1,2)$ and $\delta_\psi^3 f \in C[a,T]$. On the uniform partition with $t_k = t_0 + j\tau$ and $t_{k-\frac{1}{2}} = t_{k-1} + \frac{1}{2}\tau$ for $1 \le k \le N$, the truncation error $R^{k-\frac{1}{2}}$ satisfies the following bounds for all sufficiently small τ:

$$\left|R^{k-\frac{1}{2}}\right| \le \frac{1}{\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_\psi^3 f(t)\right| \left(\psi(t_1) - \psi(t_0)\right) \left(\psi(t_{\frac{1}{2}}) - \psi(t_0)\right)^{2-\alpha}, \quad k = 1,$$

$$\begin{aligned}\left|R^{k-\frac{1}{2}}\right| \le & \frac{1}{\Gamma(2-\alpha)} \left\{ \frac{\psi(T) - \psi(a)}{6} \left(1 + \frac{5}{4} \max_{1 \le l \le k-2} \frac{|\psi''(t_l)|}{(\psi'(t_l))^2}\right) + \frac{8}{3} \right\} \\ & \times \max_{t_0 \le t \le t_{k-1}} \left|\delta_\psi^3 f(t)\right| \max_{1 \le l \le k-1} \left(\psi(t_l) - \psi(t_{l-1})\right)^2 \left(\psi(t_{k-\frac{1}{2}}) - \psi(t_{k-\frac{3}{2}})\right)^{1-\alpha} \\ & + \frac{4}{3\Gamma(3-\alpha)} \max_{t_{k-2} \le t \le t_k} \left|\delta_\psi^3 f(t)\right| \max_{k-1 \le l \le k} \left(\psi(t_l) - \psi(t_{l-1})\right) \\ & \times \left(\psi(t_{k-\frac{1}{2}}) - \psi(t_{k-\frac{3}{2}})\right)^{2-\alpha}, \quad k \ge 2;\end{aligned}$$

H2N2 discretisation for **Type A**

Theorem

That is,

$$\left|R^{k-\frac{1}{2}}\right| \le C\tau^{3-\alpha},$$

where

$$C = \frac{1}{\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta_\psi^3 f(t)\right| \psi'(\xi_1) \left(\frac{1}{2}\psi'(\eta_0)\right)^{2-\alpha}, \quad k=1,$$

$$\begin{aligned} C =& \frac{1}{\Gamma(2-\alpha)} \left\{ \frac{\psi(T)-\psi(a)}{6} \left(1 + \frac{5}{4} \max_{1\le l\le k-2} \frac{|\psi''(t_l)|}{(\psi'(t_l))^2}\right) + \frac{8}{3} \right\} \\ & \times \max_{t_0 \le t \le t_{k-1}} \left|\delta_\psi^3 f(t)\right| \max_{1\le l\le k-1} \left(\psi'(\xi_l)\right)^2 \left(\psi'(\eta_{k-1})\right)^{1-\alpha} \\ & + \frac{4}{3\Gamma(3-\alpha)} \max_{t_{k-2}\le t\le t_k} \left|\delta_\psi^3 f(t)\right| \max_{k-1\le l\le k} \left(\psi'(\xi_l)\right) \left(\psi'(\eta_{k-1})\right)^{2-\alpha}, \quad k \ge 2, \end{aligned}$$

with $\eta_0 \in (t_0, t_{\frac{1}{2}})$, $\eta_{k-1} \in (t_{k-\frac{3}{2}}, t_{k-\frac{1}{2}})$, and $\xi_l \in (t_{l-1}, t_l)$ for $l = 1, 2, \cdots, k$.

H2N2 discretisation for **Type A**

Theorem

Let $\alpha \in (1,2)$ and assume that $\psi \in C^1[a,T]$. For all sufficiently small $\tau = t_k - t_{k-1} = \dfrac{T-a}{N}$, the coefficients $c_{j,k}^{(\alpha)}$ for $1 \le j \le k \le N$ satisfy

$$c_{j,k}^{(\alpha)} < 0 \ \text{ for } 2 \le j \le k-1, \ c_{k,k}^{(\alpha)} > 0.$$

Theorem

Let $\alpha \in (1,2)$. Assume that $\tau = t_k - t_{k-1} = \dfrac{T-a}{N}$ is sufficiently small. Assume also that $\psi \in C^2[a,T]$, $\psi' \ge M > 0$ for a constant M, and ψ''' exists on $[a,T]$. Then for $1 \le j \le k \le N$, the coefficients $c_{j,k}^{(\alpha)}$ satisfy

(1) $c_{j+1,k}^{(\alpha)} < c_{j,k}^{(\alpha)}$ for $2 \le j \le k-2$; (2) $\left|c_{k-1,k}^{(\alpha)}\right| < c_{k,k}^{(\alpha)}$ for $k \ge 2$.

H2N2 discretisation for Type A

Table: The coefficients $c_{j,k}^{(\alpha)}$ of H2N2 discretisation

$k=1$	$c_{1,1}^{(\alpha)}$					
$k=2$	$\boxed{c_{1,2}^{(\alpha)}}$	$c_{2,2}^{(\alpha)}$				
$k=3$	$\boxed{c_{1,3}^{(\alpha)}}$	$c_{2,3}^{(\alpha)}$	$c_{3,3}^{(\alpha)}$			
$k=4$	$\boxed{c_{1,4}^{(\alpha)}}$	$c_{2,4}^{(\alpha)}$	$c_{3,4}^{(\alpha)}$	$c_{4,4}^{(\alpha)}$		
$k=5$	$\boxed{c_{1,5}^{(\alpha)}}$	$c_{2,5}^{(\alpha)}$	$c_{3,5}^{(\alpha)}$	$c_{4,5}^{(\alpha)}$	$c_{5,5}^{(\alpha)}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

[1] The signs of the coefficients $c_{1,k}^{(\alpha)}$ in the boxes are uncertain.

[2] In row k, the 2nd to the $(k-2)$-th coefficients are strictly decreasing and $|c_{k-1,k}^{(\alpha)}| < c_{k,k}^{(\alpha)}$.

L2$_1$ discretisation for **Type A**

Setting $g(t)=\delta_\psi f(t)$, then ${}_{C\psi}\mathrm{D}^{\alpha}_{a,t}f(t)={}_{C\psi}\mathrm{D}^{\beta}_{a,t}g(t)$ where $\beta=\alpha-1\in(0,1)$. Set $t^*_{k-\frac{1}{2}}=\psi^{-1}\left(\frac{\psi(t_k)+\psi(t_{k-1})}{2}\right)$ for $1\le k\le N$, that is, $\psi(t^*_{k-\frac{1}{2}})=\frac{1}{2}\left(\psi(t_k)+\psi(t_{k-1})\right)$. Using linear Lagrange interpolation in the sense of the function $\psi(t)$, one has

$$
\begin{aligned}
{}_{C\psi}\mathrm{D}^{\alpha}_{a,t}f(t)\Big|_{t=t^*_{k-\frac{1}{2}}} &={}_{C\psi}\ \mathrm{D}^{\beta}_{a,t}g(t)\Big|_{t=t^*_{k-\frac{1}{2}}}\\
&=\frac{1}{\Gamma(1-\beta)}\left\{\frac{g(t^*_{\frac{1}{2}})-g(t_0)}{\psi(t^*_{\frac{1}{2}})-\psi(t_0)}\int_{t_0}^{t^*_{\frac{1}{2}}}\left(\psi(t^*_{k-\frac{1}{2}})-\psi(s)\right)^{-\beta}\psi'(s)\mathrm{d}s\right.\\
&\left.+\sum_{j=1}^{k-1}\frac{g(t^*_{j+\frac{1}{2}})-g(t^*_{j-\frac{1}{2}})}{\psi(t^*_{j+\frac{1}{2}})-\psi(t^*_{j-\frac{1}{2}})}\int_{t^*_{j-\frac{1}{2}}}^{t^*_{j+\frac{1}{2}}}\left(\psi(t^*_{k-\frac{1}{2}})-\psi(s)\right)^{-\beta}\psi'(s)\mathrm{d}s\right\}+\Upsilon^{k-\frac{1}{2}},
\end{aligned}
$$

where

$$
\begin{aligned}
\Upsilon^{k-\frac{1}{2}}=\frac{1}{\Gamma(1-\beta)}&\left\{\int_{t_0}^{t^*_{\frac{1}{2}}}\left(\psi(t^*_{k-\frac{1}{2}})-\psi(s)\right)^{-\beta}\left(\delta_\psi g(s)-\frac{g(t^*_{\frac{1}{2}})-g(t_0)}{\psi(t^*_{\frac{1}{2}})-\psi(t_0)}\right)\psi'(s)\mathrm{d}s\right.\\
&\left.+\sum_{j=1}^{k-1}\int_{t^*_{j-\frac{1}{2}}}^{t^*_{j+\frac{1}{2}}}\left(\psi(t^*_{k-\frac{1}{2}})-\psi(s)\right)^{-\beta}\left(\delta_\psi g(s)-\frac{g(t^*_{j+\frac{1}{2}})-g(t^*_{j-\frac{1}{2}})}{\psi(t^*_{j+\frac{1}{2}})-\psi(t^*_{j-\frac{1}{2}})}\right)\psi'(s)\mathrm{d}s\right\}.
\end{aligned}
$$

L2$_1$ discretisation for Type A

We define the L2$_1$ discretisation of the ψ-Caputo fractional derivative at $t = t^*_{k-\frac{1}{2}}$ by

$$
{}_{C\psi}\mathcal{D}^{\alpha}_{a,t} f^{k-\frac{1}{2}} := \frac{1}{\Gamma(3-\alpha)} \Bigg\{ a^{(\alpha)}_{k-1,k} \nabla_{\psi,t} f^{k-\frac{1}{2}} - \sum_{j=1}^{k-1} \left(a^{(\alpha)}_{j,k} - a^{(\alpha)}_{j-1,k} \right) \nabla_{\psi,t} f^{j-\frac{1}{2}} - a^{(\alpha)}_{0,k} \delta_\psi f(t_0) \Bigg\},
$$

where

$$
a^{(\alpha)}_{0,k} = \frac{\left(\psi(t^*_{k-\frac{1}{2}}) - \psi(t_0)\right)^{2-\alpha} - \left(\psi(t^*_{k-\frac{1}{2}}) - \psi(t^*_{\frac{1}{2}})\right)^{2-\alpha}}{\psi(t^*_{\frac{1}{2}}) - \psi(t_0)},
$$

$$
a^{(\alpha)}_{j,k} = \frac{\left(\psi(t^*_{k-\frac{1}{2}}) - \psi(t^*_{j-\frac{1}{2}})\right)^{2-\alpha} - \left(\psi(t^*_{k-\frac{1}{2}}) - \psi(t^*_{j+\frac{1}{2}})\right)^{2-\alpha}}{\psi(t^*_{j+\frac{1}{2}}) - \psi(t^*_{j-\frac{1}{2}})}, \quad 1 \le j \le k-1.
$$

L2_1 discretisation for **Type A**

The truncation error $R^{k-\frac{1}{2}}$ of the L2_1 discretisation is

$$R^{k-\frac{1}{2}} = {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t)\Big|_{t=t^*_{k-\frac{1}{2}}} - {}_{C\psi}\mathcal{D}^{\alpha}_{a,t} f^{k-\frac{1}{2}} = \Upsilon^{k-\frac{1}{2}} + r^{k-\frac{1}{2}},$$

where $\Upsilon^{k-\frac{1}{2}}$ is defined above and

$$r^{k-\frac{1}{2}} = \frac{1}{\Gamma(3-\alpha)} \Bigg\{ a^{(\alpha)}_{k-1,k} \left(\delta_\psi f(t^*_{k-\frac{1}{2}}) - \nabla_{\psi,t} f^{k-\frac{1}{2}} \right) - \sum_{j=1}^{k-1} \left(a^{(\alpha)}_{j,k} - a^{(\alpha)}_{j-1,k} \right) \left(\delta_\psi f(t^*_{j-\frac{1}{2}}) - \nabla_{\psi,t} f^{j-\frac{1}{2}} \right) \Bigg\}.$$

L2$_1$ discretisation for **Type A**

Theorem

Let $\delta_\psi^3 f \in C[a,T]$ and $\alpha \in (1,2)$. The truncation errors $R^{k-\frac{1}{2}}$ $(1 \le k \le N)$ satisfy the estimate

$$\left|R^{k-\frac{1}{2}}\right| \le \left[\frac{2^{\alpha-1}}{4\Gamma(2-\alpha)} + \frac{6+2^{\alpha-1}}{12\Gamma(3-\alpha)}\right] \max_{t_0\le t\le t_k} \left|\delta_\psi^3 f(t)\right| \max_{1\le l\le k} \big(\psi(t_l)-\psi(t_{l-1})\big)^{3-\alpha};$$

that is,

$$\left|R^{k-\frac{1}{2}}\right| \le C\tau^{3-\alpha},$$

where

$$C = \left[\frac{2^{\alpha-1}}{4\Gamma(2-\alpha)} + \frac{6+2^{\alpha-1}}{12\Gamma(3-\alpha)}\right] \max_{t_0\le t\le t_k} \left|\delta_\psi^3 f(t)\right| \max_{1\le l\le k} \big(\psi'(\xi_l)\big)^{3-\alpha},$$

with $\xi_l \in (t_{l-1}, t_l)$.

L2$_1$ discretisation for **Type A**

Theorem

Let $\alpha \in (1,2)$. For $1 \leq k \leq N$, the coefficients $a_{j,k}^{(\alpha)}$ of the L2$_1$ discretisation satisfy

$$a_{k-1,k}^{(\alpha)} > a_{k-2,k}^{(\alpha)} > \cdots > a_{0,k}^{(\alpha)} > 0.$$

Relationship with Riemann-Liouville fractional calculus

Lemma

For $\alpha > 0$ and $x \in L^1(a,T)$, let $\widetilde{s} = \psi(s) - \psi(a)$, then

$$\begin{aligned} {}_{\psi}\mathrm{D}_{a,t}^{-\alpha} x(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (\psi(t) - \psi(s))^{\alpha-1} x(s) \psi'(s) \mathrm{d}s \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{t'} (t' - \tilde{s})^{\alpha-1} \tilde{x}_a(\tilde{s}) \mathrm{d}\tilde{s} \\ &= {}_{RL}\mathrm{D}_{0,t'}^{-\alpha} \tilde{x}_a(t'), \end{aligned}$$

where $t' = \psi(t) - \psi(a)$, $\widetilde{x}_a(\widetilde{s}) = x\left(\psi^{-1}(\widetilde{s} + \psi(a))\right) = x(s)$.

Relationship with Riemann-Liouville fractional calculus

Lemma

For $n-1<\alpha<n\in\mathbb{Z}^+$ and $x(t)\in AC^n_{\delta_\psi}[a,T]$, let $\widetilde{s}=\psi(s)-\psi(a)$, then

$$
\begin{aligned}
{}_{C\psi}\mathrm{D}^{\alpha}_{a,t}x(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (\psi(t)-\psi(s))^{n-\alpha-1}\delta_\psi^n x(s)\psi'(s)\mathrm{d}s \\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^{t'} (t'-\tilde{s})^{n-\alpha-1}\tilde{x}_a^{(n)}(\tilde{s})\mathrm{d}\tilde{s} \\
&= {}_{C}\mathrm{D}^{\alpha}_{0,t'}\tilde{x}_a(t'),
\end{aligned}
$$

where $t'=\psi(t)-\psi(a)$, $\widetilde{x}_a(\widetilde{s})=x\left(\psi^{-1}(\widetilde{s}+\psi(a))\right)=x(s)$.

Relationship with Riemann-Liouville fractional calculus

Lemma

For $n-1<\alpha<n\in\mathbb{Z}^+$ and $x(t)\in AC^n_{\delta_\psi}[a,T]$, let $\widetilde{s}=\psi(s)-\psi(a)$, then

$$\begin{aligned}{}_{\psi}\mathrm{D}^{\alpha}_{a,t}x(t) &= \frac{1}{\Gamma(n-\alpha)}\delta^n_\psi\int_a^t(\psi(t)-\psi(s))^{n-\alpha-1}x(s)\psi'(s)\mathrm{d}s\\ &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}(t')^n}\int_0^{t'}(t'-\tilde{s})^{n-\alpha-1}\tilde{x}_a(\tilde{s})\mathrm{d}\tilde{s}\\ &= {}_{RL}\mathrm{D}^{\alpha}_{0,t'}\tilde{x}_a(t'),\end{aligned}$$

where $t'=\psi(t)-\psi(a)$, $\widetilde{x}_a(\widetilde{s})=x\left(\psi^{-1}(\widetilde{s}+\psi(a))\right)=x(s)$.

Relationship with Riemann-Liouville fractional calculus

The right-hand-side function $g(t,x)$ in the differential equations (3.1) and (3.5) below is assumed to be continuous in the given domain and to satisfy a Lipschitz condition in the second variable; these conditions guarantee existence and uniqueness of the solutions to these differential equations.

Lemma

For $n-1<\alpha<n\in\mathbb{Z}^+$ and $x\in AC^n_{\delta_\psi}[a,T]$, the nonlinear ψ-Caputo fractional derivative initial-value problem

$$\begin{cases} {}_{C\psi}\mathrm{D}^{\alpha}_{a,t}x(t)=g\left(t,x\left(t\right)\right),\ a<t\leq T,\\ \delta^k_\psi x(a)=x_{ak},\ k=0,1,\cdots,n-1, \end{cases} \tag{3.1}$$

is equivalent to the nonlinear Caputo fractional derivative initial-value problem

$$\begin{cases} {}_{C}\mathrm{D}^{\alpha}_{0,t'}\widetilde{x}_a(t')=g\left(\psi^{-1}(t'+\psi(a)),\widetilde{x}_a\left(t'\right)\right),\ 0<t'\leq\psi(T)-\psi(a),\\ \widetilde{x}^{(k)}_a(0)=x_{ak},\ k=0,1,\cdots,n-1. \end{cases} \tag{3.2}$$

Relationship with Riemann-Liouville fractional calculus

Remark

Let $\alpha \in (n-1, n)$ with $n \in \mathbb{Z}^+$. From the definition of ψ-Caputo fractional derivative, one can see that (3.1) is equivalent to the Volterra integral equation

$$x(t) = \sum_{k=0}^{n-1} \frac{(\psi(t)-\psi(a))^k}{k!} x_{ak} + \frac{1}{\Gamma(\alpha)} \int_a^t (\psi(t)-\psi(s))^{\alpha-1} g(s, x(s)) \psi'(s) \mathrm{d}s \quad (3.3)$$

provided that $x \in AC^n_{\delta_\psi}[a, T]$. Similarly, (3.2) is equivalent to the Volterra integral equation

$$\widetilde{x}_a(t') = \sum_{k=0}^{n-1} \frac{(t')^k}{k!} x_{ak} + \frac{1}{\Gamma(\alpha)} \int_0^{t'} (t'-s)^{\alpha-1} g(\psi^{-1}(s+\psi(a)), \widetilde{x}_a(s)) \mathrm{d}s. \quad (3.4)$$

provided that $\widetilde{x} \in AC^n[0, \psi(T)-\psi(a)]$. (Note that when using the α-order Caputo fractional derivative of a given function v, in the literature the condition that $v \in AC^n$ is often assumed without being stated.)

One can see easily by making the substitution $t' = \psi(t) - \psi(a)$ that the Volterra equations (3.3) and (3.4) are equivalent and their solutions satisfy the relation $x(t) = x\left(\psi^{-1}(t'+\psi(a))\right) = \widetilde{x}_a(t')$.

Relationship with Riemann-Liouville fractional calculus

Remark

The above lemma does not mean that the ψ-Caputo derivative is not needed. One may require it, for example, in either of the following two situations:

(1) The inverse ψ^{-1}, which we know exists, may not be available explicitly even though $\psi(t)$ is known explicitly. For example if $\psi(t) = te^t$, then its inverse ψ^{-1} cannot be found analytically.

(2) The right-hand-side of (3.2), which is obtained from the right-hand-side of (3.1) via the transformation $t' = \psi(t) - \psi(a)$, may lose regularity. For example, if $a = -1$, $\psi(t) = t^3 + \beta t$ $(\beta > 0)$ and $g(t, x(t)) = t$, then $\partial g/\partial t = 1$ which is well behaved, but transforming using $t' = t^3 + \beta t + 1 + \beta$ gives

$$g\left(\psi^{-1}(t' + \psi(a))\right) = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}},$$

where $p = \beta$, $q = 1 + \beta - t'$, $t' \geq 0$, and the first-order derivative of this function with respect to t' can be very large for any fixed $t' > 0$ as $\beta \to 0^+$.

Relationship with Riemann-Liouville fractional calculus

Lemma

For $n-1<\alpha<n\in\mathbb{Z}^+$ and $x\in AC^n_{\delta_\psi}[a,T]$, the nonlinear ψ-Riemann-Liouville fractional derivative initial-value problem

$$\begin{cases} {}_\psi\mathrm{D}^{\alpha}_{a,t}x(t)=g\left(t,x\left(t\right)\right),\ a<t<T, \\ {}_\psi\mathrm{D}^{\alpha+k-n}_{a,t}x(a)=x_{ak},\ k=0,1,\cdots,n-1, \end{cases} \tag{3.5}$$

is equivalent to the nonlinear Riemann-Liouville fractional derivative initial-value problem

$$\begin{cases} {}_{RL}\mathrm{D}^{\alpha}_{0,t'}\widetilde{x}_a(t')=g\left(\psi^{-1}(t'+\psi(a)),\widetilde{x}_a\left(t'\right)\right),\ 0<t'<\psi(T)-\psi(a), \\ {}_{RL}\mathrm{D}^{\alpha+k-n}_{0,t'}\widetilde{x}_a(0)=x_{ak},\ k=0,1,\cdots,n-1. \end{cases} \tag{3.6}$$

Relationship with Riemann-Liouville fractional calculus

Remark

From the definition of the ψ-Riemann-Liouville fractional derivative, one can see that (3.5) is equivalent to the Volterra integral equation

$$x(t) = \sum_{k=0}^{n-1} \frac{(\psi(t)-\psi(a))^{\alpha-k-1}}{\Gamma(\alpha-k)} x_{ak} + \frac{1}{\Gamma(\alpha)} \int_a^t (\psi(t)-\psi(s))^{\alpha-1} g(s, x(s)) \psi'(s) \mathrm{d}s. \tag{3.7}$$

provided that $x \in AC^n_{\delta_\psi}[a,T]$. Similarly, (3.6) is equivalent to the Volterra integral equation

$$\widetilde{x}_a(t') = \sum_{k=0}^{n-1} \frac{(t')^{\alpha-k-1}}{\Gamma(\alpha-k)} x_{ak} + \frac{1}{\Gamma(\alpha)} \int_0^{t'} (t'-s)^{\alpha-1} g(\psi^{-1}(s+\psi(a)), \widetilde{x}_a(s)) \mathrm{d}s. \tag{3.8}$$

provided that $\widetilde{x} \in AC^n\,[0, \psi(T)-\psi(a)]$. (Note that when using the α-order Riemann-Liouville fractional derivative of a given function v, in the literature the condition that $v \in AC^n$ is often assumed without being stated.) One can see easily by making the substitution $t' = \psi(t) - \psi(a)$ that the Volterra equations (3.7) and (3.8) are equivalent and their solutions satisfy the relation $x(t) = x\left(\psi^{-1}(t'+\psi(a))\right) = \widetilde{x}_a(t')$.

L1 discretisation for $\mathbf{Type\ B}$

If one takes the special non-uniform $\mathbf{Type\ B}$ partition with $\widetilde{\tau} = \dfrac{\psi(T) - \psi(a)}{N}$, the L1 discretisation becomes

$$ {}_{C\psi}\mathscr{D}^{\alpha}_{a,t} f^k = \frac{1}{\Gamma(2-\alpha)} \left\{ \widetilde{a}^{(\alpha)}_{k,k} f^k - \sum_{j=1}^{k-1} \left(\widetilde{a}^{(\alpha)}_{j+1,k} - \widetilde{a}^{(\alpha)}_{j,k} \right) f^j - \widetilde{a}^{(\alpha)}_{1,k} f^0 \right\}, $$

where

$$ \widetilde{a}^{(\alpha)}_{j,k} = \widetilde{\tau}^{-\alpha} \left[(k-j+1)^{1-\alpha} - (k-j)^{1-\alpha} \right], \quad j = 1, 2, \cdots, k. $$

L1 discretisation for **Type B**

Theorem

For $0 < \alpha < 1$ and $\delta_\psi^2 f(t) \in C[a,T]$, the truncation error of the L1 discretisation is bounded by

$$\left|R^k\right| \leq \left\{\frac{1}{8\Gamma(1-\alpha)} + \frac{\alpha}{2\Gamma(3-\alpha)}\right\} \max_{t_0\leq t\leq t_k} \left|\delta_\psi^2 f(t)\right| \widetilde{\tau}^{2-\alpha},$$

where R^k is almost the same as the case of **Type A** in form.

L1 discretisation for Type B

Theorem

The coefficients $\widetilde{a}_{j,k}^{(\alpha)}$ satisfy

$$\widetilde{a}_{k,k}^{(\alpha)} > \widetilde{a}_{k-1,k}^{(\alpha)} > \cdots > \widetilde{a}_{1,k}^{(\alpha)} > 0 \quad \text{for } 1 \leq j \leq k \leq N.$$

L1-2 discretisation for **Type B**

If the special non-uniform **Type B** partition is used, then the L1-2 discretisation analogous to the case of **Type A** can be expressed as

$$
{}_{C\psi}\mathbb{D}^{\alpha}_{a,t} f^k = \frac{1}{\Gamma(2-\alpha)} \left\{ \widetilde{c}^{(\alpha)}_{k,k} f^k - \sum_{j=1}^{k-1} \left(\widetilde{c}^{(\alpha)}_{j+1,k} - \widetilde{c}^{(\alpha)}_{j,k} \right) f^j - \widetilde{c}^{(\alpha)}_{1,k} f^0 \right\},
$$

where

$$
\widetilde{c}^{(\alpha)}_{j,k} = \begin{cases} \widetilde{\tau}^{-1} \left(\widetilde{a}^{(\alpha)}_{1,k} - \widetilde{b}^{(\alpha)}_{2,k} \right), & j = 1, \\ \widetilde{\tau}^{-1} \left(\widetilde{a}^{(\alpha)}_{j,k} + \widetilde{b}^{(\alpha)}_{j,k} - \widetilde{b}^{(\alpha)}_{j+1,k} \right), & 2 \le j \le k-1, \\ \widetilde{\tau}^{-1} \left(\widetilde{a}^{(\alpha)}_{k,k} + \widetilde{b}^{(\alpha)}_{k,k} \right), & j = k, \end{cases}
$$

$$
\widetilde{a}^{(\alpha)}_{j,k} = \widetilde{\tau}^{1-\alpha} \left[(k-j+1)^{1-\alpha} - (k-j)^{1-\alpha} \right], \quad 1 \le j \le k,
$$

$$
\widetilde{b}^{(\alpha)}_{j,k} = \widetilde{\tau}^{1-\alpha} \left\{ \frac{1}{2-\alpha} \left[(k-j+1)^{2-\alpha} - (k-j)^{2-\alpha} \right] - \frac{1}{2} \left[(k-j+1)^{1-\alpha} + (k-j)^{1-\alpha} \right] \right\}, \quad 2 \le j \le k.
$$

L1-2 discretisation for Type B

Theorem

The truncation errors $R^k\left(1 \leq k \leq N\right)$ of the L1-2 discretisation can be similarly bounded as follows:

$$\left|R^1\right| \leq \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_0 \leq t \leq t_1} \left|\delta_\psi^2 f(t)\right| \widetilde{\tau}^{2-\alpha}, \quad k=1,$$

$$\left|R^k\right| \leq \frac{\alpha}{8\Gamma(1-\alpha)} \max_{t_0 \leq t \leq t_1} \left|\delta_\psi^2 f(t)\right| \left(\psi(t_k)-\psi(t_1)\right)^{-1-\alpha} \widetilde{\tau}^3$$

$$+ \left\{\frac{1}{12\Gamma(1-\alpha)} + \frac{\alpha}{3\Gamma(2-\alpha)}\right\} \max_{t_0 \leq t \leq t_k} \left|\delta_\psi^3 f(t)\right| \widetilde{\tau}^{3-\alpha}, \quad k \geq 2.$$

L1-2 discretisation for $\mathbf{Type\ B}$

Theorem

For $1 \leq j \leq k \leq N$*, the coefficients* $\widetilde{c}_{j,k}^{(\alpha)}$ *have the following properties:*

(i) $\widetilde{c}_{j,k}^{(\alpha)} > 0, \ j \neq k-1$; (ii) The sign of $\widetilde{c}_{k-1,k}^{(\alpha)}$ is uncertain for $k \geq 2$;

(iii) $\widetilde{c}_{k-2,k}^{(\alpha)} > \widetilde{c}_{k-3,k}^{(\alpha)} > \cdots > \widetilde{c}_{1,k}^{(\alpha)}$; (iv) $\widetilde{c}_{k,k}^{(\alpha)} > \left|\widetilde{c}_{k-1,k}^{(\alpha)}\right|$ for k ≥ 2;

(v) $\widetilde{c}_{k,k}^{(\alpha)} > \widetilde{c}_{k-2,k}^{(\alpha)}$.

L2-1$_\sigma$ discretisation for **Type B**

Let $\alpha \in (0,1)$ and $\sigma = 1 - \frac{\alpha}{2}$. Consider a **Type B** non-uniform partition. Then the L2-1$_\sigma$ discretisation at $t = t^*_{k+\sigma}$ can be shown to have the following form,

$$_{C\psi}\mathfrak{D}^{\alpha}_{a,t} f^{k+\sigma} = \frac{1}{\Gamma(2-\alpha)} \left\{ \widetilde{c}^{(\alpha,\sigma)}_{k+1,k} f^{k+1} - \sum_{j=1}^{k} \left(\widetilde{c}^{(\alpha,\sigma)}_{j+1,k} - \widetilde{c}^{(\alpha,\sigma)}_{j,k} \right) f^j - \widetilde{c}^{(\alpha,\sigma)}_{1,k} f^0 \right\},$$

where

$$\widetilde{c}^{(\alpha,\sigma)}_{j,k} = \begin{cases} \widetilde{\tau}^{-1} \left(\widetilde{a}^{(\alpha,\sigma)}_{1,k} - \widetilde{b}^{(\alpha,\sigma)}_{1,k} \right), & j = 1, \\ \widetilde{\tau}^{-1} \left(\widetilde{a}^{(\alpha,\sigma)}_{j,k} + \widetilde{b}^{(\alpha,\sigma)}_{j-1,k} - \widetilde{b}^{(\alpha,\sigma)}_{j,k} \right), & 2 \leq j \leq k, \\ \widetilde{\tau}^{-1} \left(\widetilde{a}^{(\alpha,\sigma)}_{k+1,k} + \widetilde{b}^{(\alpha,\sigma)}_{k,k} \right), & j = k+1, \end{cases}$$

$$\widetilde{a}^{(\alpha,\sigma)}_{j,k} = \begin{cases} \widetilde{\tau}^{1-\alpha} \left[(k+\sigma-j+1)^{1-\alpha} - (k+\sigma-j)^{1-\alpha} \right], & 1 \leq j \leq k, \\ \widetilde{\tau}^{1-\alpha} \sigma^{1-\alpha}, & j = k+1. \end{cases}$$

$$\widetilde{b}^{(\alpha,\sigma)}_{j,k} = \widetilde{\tau}^{1-\alpha} \left\{ \frac{1}{2-\alpha} \left[(k+\sigma-j+1)^{2-\alpha} - (k+\sigma-j)^{2-\alpha} \right] - \frac{1}{2} \left[(k+\sigma-j+1)^{1-\alpha} + (k+\sigma-j)^{1-\alpha} \right] \right\}, \quad 1 \leq j \leq k.$$

L2-1$_\sigma$ discretisation for Type B

Theorem

The truncation error of L2-1$_\sigma$ discretisation for **Type B** satisfies

$$\left|R^{k+\sigma}\right| \le \left\{\frac{\sigma^{-\alpha}}{12\Gamma(1-\alpha)} + \frac{\sigma^{1-\alpha}}{6\Gamma(2-\alpha)}\right\} \max_{t_0\le t\le t_{k+1}} \left|\delta_\psi^2 f(t)\right| \widetilde{\tau}^{3-\alpha}.$$

Theorem

For $1 \le j \le k+1$ and $0 \le k \le N-1$, the coefficients $\widetilde{c}_{j,k}^{(\alpha,\sigma)}$ have the following property:

$$0 < \widetilde{c}_{1,k}^{(\alpha,\sigma)} < \widetilde{c}_{2,k}^{(\alpha,\sigma)} < \cdots < \widetilde{c}_{k,k}^{(\alpha,\sigma)} < \widetilde{c}_{k+1,k}^{(\alpha,\sigma)}.$$

L2 discretisation for Type B

Choose a uniform partition of **Type B** in the sense of the function $\psi(t)$, with $\widetilde{\tau} = \psi(t_k) - \psi(t_{k-1}) = \dfrac{\psi(T) - \psi(a)}{N}$. Then the L2 discretisation at $t = t_k$ for $1 \leq k \leq N-1$ is as follows:

$$ {}_{C\psi}\mathbf{D}^{\alpha}_{a,t} f^k = \frac{2}{\Gamma(3-\alpha)} \sum_{j=1}^{k+1} \widetilde{c}^{(\alpha)}_{j,k} \left(f^j - f^{j-1} \right), $$

where

$$ \widetilde{c}^{(\alpha)}_{j,k} = \begin{cases} -\widetilde{\tau}^{-1} \widetilde{a}^{(\alpha)}_{1,k}, & j = 1, \\ \widetilde{\tau}^{-1} \left(\widetilde{a}^{(\alpha)}_{j-1,k} - \widetilde{a}^{(\alpha)}_{j,k} \right), & 2 \leq j \leq k, \\ \widetilde{\tau}^{-1} \widetilde{a}^{(\alpha)}_{k,k}, & j = k+1, \end{cases} $$

$$ \widetilde{a}^{(\alpha)}_{j,k} = \widetilde{\tau}^{1-\alpha} \left[(k-j+1)^{2-\alpha} - (k-j)^{2-\alpha} \right], \quad 1 \leq j \leq k. $$

L2 discretisation for Type B

Theorem

The truncation error of L2 discretisation for **Type B** can be bounded

$$\left|R^k\right| \le \frac{5\left(\psi(T)-\psi(a)\right)^{2-\alpha}}{3\Gamma(3-\alpha)} \max_{t_0\le t\le t_{k+1}} \left|\delta_\psi^3 f(t)\right| \widetilde{\tau}.$$

Theorem

The coefficients $\widetilde{c}_{j,k}^{(\alpha)}$ have the following properties for $1 \le j \le k \le N$

(i) $\widetilde{c}_{k+1,k}^{(\alpha)} > 0,\ \widetilde{c}_{j,k}^{(\alpha)} < 0$ for $1 \le j \le k$, (ii) $\left|\widetilde{c}_{k,k}^{(\alpha)}\right| < \widetilde{c}_{k+1,k}^{(\alpha)}$ for $k \ge 2$,

(iii) $\widetilde{c}_{2,k}^{(\alpha)} > \widetilde{c}_{3,k}^{(\alpha)} > \cdots > \widetilde{c}_{k,k}^{(\alpha)},\ \widetilde{c}_{1,k}^{(\alpha)} < \widetilde{c}_{2,k}^{(\alpha)}$ for $k \ge 3$.

H2N2 discretisation for $\mathbf{Type\ B}$

If a $\mathbf{Type\ B}$ partition is used, then the H2N2 formula at $t = t^*_{k-\frac{1}{2}} = \psi^{-1}[\frac{1}{2}\psi(t_k) + \frac{1}{2}\psi(t_{k-1})]$ for $1 \le k \le N$ is as follows,

$$ {}_{C\psi}\mathbb{D}^{\alpha}_{a,t} f^{k-\frac{1}{2}} = \frac{2}{\Gamma(3-\alpha)} \sum_{j=1}^{k} \widetilde{c}^{(\alpha)}_{j,k} \left(f^j - f^{j-1} \right) - \frac{2}{\Gamma(3-\alpha)} \widetilde{a}^{(\alpha)}_{0,k}\, \delta_\psi f(t_0), $$

where

$$ \widetilde{c}^{(\alpha)}_{j,k} = \begin{cases} \widetilde{\tau}^{-1} \left(\widetilde{a}^{(\alpha)}_{j-1,k} - \widetilde{a}^{(\alpha)}_{j,k} \right), & 1 \le j \le k-1, \\ \widetilde{\tau}^{-1} \widetilde{a}^{(\alpha)}_{k-1,k}, & j = k, \end{cases} $$

$$ \widetilde{a}^{(\alpha)}_{j,k} = \begin{cases} \widetilde{\tau}^{1-\alpha} \left[(k-\frac{1}{2})^{2-\alpha} - (k-1)^{2-\alpha} \right], & j = 0, \\ \frac{\widetilde{\tau}^{1-\alpha}}{2} \left[(k-j)^{2-\alpha} - (k-j-1)^{2-\alpha} \right], & 1 \le j \le k-1. \end{cases} $$

H2N2 discretisation for **Type B**

Theorem

The truncation error of H2N2 discretisation for **Type B** can be bounded as follows

$$\left|R^{k-\frac{1}{2}}\right| \le \frac{1}{2^{2-\alpha}\Gamma(3-\alpha)} \max_{t_0\le t\le t_1} \left|\delta_\psi^3 f(t)\right| \widetilde{\tau}^{3-\alpha}, \quad k=1,$$

$$\left|R^{k-\frac{1}{2}}\right| \le \left\{\frac{2}{\Gamma(2-\alpha)} + \frac{4}{3\Gamma(3-\alpha)}\right\} \max_{t_0\le t\le t_k} \left|\delta_\psi^3 f(t)\right| \widetilde{\tau}^{3-\alpha}, \quad k\ge 2.$$

Theorem

The coefficients $\widetilde{c}_{j,k}^{(\alpha)}$ for $1\le j\le k\le N$ have the following properties

(i) $\widetilde{c}_{k,k}^{(\alpha)} > 0, \quad \widetilde{c}_{j,k}^{(\alpha)} < 0$ for $1\le j\le k-1$,

(ii) $\widetilde{c}_{1,k}^{(\alpha)} > \widetilde{c}_{2,k}^{(\alpha)} > \cdots > \widetilde{c}_{k-1,k}^{(\alpha)}$ for $k\ge 3$, (iii) $\left|\widetilde{c}_{k-1,k}^{(\alpha)}\right| < \widetilde{c}_{k,k}^{(\alpha)}$.

L2$_1$ discretisation for Type B

Suppose we use the special **Type B** non-uniform partition. The L2$_1$ discretisation at $t = t^*_{k-\frac{1}{2}}$ $(1 \le k \le N)$ is

$$
{}_{C\psi}\mathcal{D}^{\alpha}_{a,t} f^{k-\frac{1}{2}} = \frac{1}{\Gamma(3-\alpha)} \Big\{ \widetilde{a}^{(\alpha)}_{k-1,k} \nabla_{\psi,t} f^{k-\frac{1}{2}} - \sum_{j=1}^{k-1} \left(\widetilde{a}^{(\alpha)}_{j,k} - \widetilde{a}^{(\alpha)}_{j-1,k} \right) \nabla_{\psi,t} f^{j-\frac{1}{2}} - \widetilde{a}^{(\alpha)}_{0,k} \delta_\psi f(t_0) \Big\},
$$

where

$$
\widetilde{a}^{(\alpha)}_{j,k} = \begin{cases} 2\widetilde{\tau}^{1-\alpha} \left[(k-\frac{1}{2})^{2-\alpha} - (k-1)^{2-\alpha} \right], & j = 0, \\ \widetilde{\tau}^{1-\alpha} \left[(k-j)^{2-\alpha} - (k-j-1)^{2-\alpha} \right], & 1 \le j \le k-1. \end{cases}
$$

L2$_1$ discretisation for **Type B**

Theorem

The truncation error of L2$_1$ discretisation for **Type B** can be easily estimated

$$\left|R^{k-\frac{1}{2}}\right| \leq \left\{\frac{1}{4\Gamma(2-\alpha)} + \frac{7}{12\Gamma(3-\alpha)}\right\} \max_{t_0\leq t\leq t_k} \left|\delta_\psi^3 f(t)\right| \widetilde{\tau}^{3-\alpha}.$$

Theorem

The coefficients $\widetilde{a}_{j,k}^{(\alpha)}$ have the property

$$\widetilde{a}_{k-1,k}^{(\alpha)} > \widetilde{a}_{k-2,k}^{(\alpha)} > \cdots > \widetilde{a}_{0,k}^{(\alpha)} > 0.$$

On **Type B** partition the H2N2 and L2$_1$ discretisations are essentially the same, but this is not true of these discretisations on **Type A** partition.

Numerical Examples

In this section we present numerical examples to test the convergence orders of some of the discretisations that were derived earlier. Here we consider only **Type A** partitions.

We shall examine $\psi(t) = t^\rho$ where $\rho > 0$ is constant.

Example

Let $f(t) = t^{2\rho}$, $[a, T] = [1, 2]$, $\alpha \in (0, 1)$. A simple calculation gives

$$ {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t) = \frac{2}{\Gamma(2-\alpha)} a^\rho (t^\rho - a^\rho)^{1-\alpha} + \frac{2}{\Gamma(3-\alpha)} (t^\rho - a^\rho)^{2-\alpha}. $$

Here, set Error $= \left| {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t_N) - {}_{C\psi}\mathscr{D}^{\alpha}_{a,t} f^N \right|$ for the L1 discretisation.

Numerical Examples

Table: Convergence rates with $\alpha \in (0,1)$ for L1 discretisation

	α	0.2		0.5		0.8	
ρ	N	Error	Rate	Error	Rate	Error	Rate
	200	2.4733E-06	–	3.4739E-05	–	3.7947E-04	–
$1/2$	300	1.2084E-06	1.7666	1.8922E-05	1.4984	2.3320E-04	1.2008
	400	7.2639E-07	1.7691	1.2295E-05	1.4986	1.6509E-04	1.2006
	500	4.8928E-07	1.7709	8.8001E-06	1.4988	1.2629E-04	1.2005
	200	1.7449E-04	–	1.2884E-03	–	6.9391E-03	–
2	300	8.6158E-05	1.7404	7.0523E-04	1.4864	4.2705E-03	1.1973
	400	5.2145E-05	1.7455	4.5956E-04	1.4886	3.0255E-03	1.1980
	500	3.5295E-05	1.7490	3.2956E-04	1.4901	2.3156E-03	1.1983

Numerical Examples

Example

Choose $f(t) = t^{3\rho}$, $[a, T] = [1, 2]$, $\alpha \in (0, 1)$. Then

$$ {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t) = \frac{3}{\Gamma(2-\alpha)} a^{2\rho} (t^{\rho} - a^{\rho})^{1-\alpha} + \frac{6}{\Gamma(3-\alpha)} a^{\rho} (t^{\rho} - a^{\rho})^{2-\alpha} + \frac{6}{\Gamma(4-\alpha)} (t^{\rho} - a^{\rho})^{3-\alpha}. $$

For the L1-2 discretisation, let Error $= \left| {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t_N) - {}_{C\psi}\mathbb{D}^{\alpha}_{a,t} f^{N} \right|$; and for the L2-$1_{\sigma}$ discretisation, let Error $= \left| {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t_{N-1+\sigma}) - {}_{C\psi}\mathfrak{D}^{\alpha}_{a,t} f^{N-1+\sigma} \right|$.

Numerical Examples

Table: Convergence rates with $\alpha \in (0,1)$ for L1-2 discretisation

	α	0.2		0.5		0.8	
ρ	N	Error	Rate	Error	Rate	Error	Rate
	200	1.1127E-08	–	1.1615E-07	–	1.2778E-06	–
$1/2$	300	3.5023E-09	2.8509	4.1568E-08	2.5341	5.2246E-07	2.2057
	400	1.5436E-09	2.8479	2.0082E-08	2.5288	2.7711E-07	2.2043
	500	8.1789E-10	2.8464	1.1431E-08	2.5255	1.6948E-07	2.2034
	200	5.6224E-06	–	4.4886E-05	–	2.6187E-04	–
2	300	1.8508E-06	2.7404	1.6390E-05	2.4846	1.0753E-04	2.1952
	400	8.4004E-07	2.7459	8.0132E-06	2.4874	5.7161E-05	2.1965
	500	4.5483E-07	2.7495	4.5981E-06	2.4892	3.5008E-05	2.1972

Numerical Examples

Table: Convergence rates with $\alpha \in (0, 1)$ for L2-1_σ discretisation

	α	0.2		0.5		0.8	
ρ	N	Error	Rate	Error	Rate	Error	Rate
	200	1.0465E-09	–	1.4682E-08	–	2.9491E-07	–
$1/2$	300	3.6206E-10	2.6177	5.3230E-09	2.5023	1.2091E-07	2.1990
	400	1.6924E-10	2.6435	2.5925E-09	2.5008	6.4227E-08	2.1991
	500	9.3587E-11	2.6548	1.4839E-09	2.5005	3.9318E-08	2.1992
	200	3.9780E-06	–	2.3440E-05	–	9.9503E-05	–
2	300	1.3211E-06	2.7186	8.5934E-06	2.4748	4.0883E-05	2.1937
	400	6.0308E-07	2.7259	4.2116E-06	2.4790	2.1740E-05	2.1952
	500	3.2789E-07	2.7308	2.4208E-06	2.4815	1.3318E-05	2.1961

Numerical Examples

Example

Let $f(t) = t^{3\rho}$, $[a, T] = [1, 2]$, $\alpha \in (1, 2)$. One has

$$ {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t) = \frac{6}{\Gamma(3-\alpha)} a^{\rho} (t^{\rho} - a^{\rho})^{2-\alpha} + \frac{6}{\Gamma(4-\alpha)} (t^{\rho} - a^{\rho})^{3-\alpha}. $$

Here, let Error $= \left| {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t_{N-1}) - {}_{C\psi}\mathbf{D}^{\alpha}_{a,t} f^{N-1} \right|$ for the L2 discretisation; Error $= \left| {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t_{N-\frac{1}{2}}) - {}_{C\psi}\mathbb{D}^{\alpha}_{a,t} f^{N-\frac{1}{2}} \right|$ for the H2N2 discretisation; and Error $= \left| {}_{C\psi}\mathrm{D}^{\alpha}_{a,t} f(t^{*}_{N-\frac{1}{2}}) - {}_{C\psi}\mathcal{D}^{\alpha}_{a,t} f^{N-\frac{1}{2}} \right|$ for the $\mathrm{L2}_1$ discretisation.

Numerical Examples

Table: Convergence rates with $\alpha \in (1, 2)$ for L2 discretisation

	α	1.2		1.5		1.8	
ρ	N	Error	Rate	Error	Rate	Error	Rate
	200	3.2478E-03	–	4.1969E-03	–	3.9832E-03	–
$1/2$	300	2.1693E-03	0.9953	2.8129E-03	0.9868	2.7159E-03	0.9445
	400	1.6286E-03	0.9966	2.1163E-03	0.9892	2.0668E-03	0.9494
	500	1.3036E-03	0.9973	1.6965E-03	0.9907	1.6710E-03	0.9527
	200	1.2352E-01	–	9.6004E-02	–	5.4651E-02	–
2	300	8.2585E-02	0.9930	6.4540E-02	0.9794	3.7523E-02	0.9274
	400	6.2030E-02	0.9948	4.8642E-02	0.9830	2.8683E-02	0.9339
	500	4.9670E-02	0.9959	3.9041E-02	0.9853	2.3265E-02	0.9382

Numerical Examples

Table: Convergence rates with $\alpha \in (1, 2)$ for H2N2 discretisation

	α	1.2		1.5		1.8	
ρ	N	Error	Rate	Error	Rate	Error	Rate
	200	8.1888E-06	–	1.0584E-04	–	1.1416E-03	–
$1/2$	300	3.9673E-06	1.7873	5.7482E-05	1.5056	7.0100E-04	1.2029
	400	2.3717E-06	1.7883	3.7286E-05	1.5046	4.9605E-04	1.2021
	500	1.5911E-06	1.7890	2.6656E-05	1.5040	3.7937E-04	1.2017
	200	3.4389E-04	–	3.7279E-03	–	2.0699E-02	–
2	300	1.7874E-04	1.6140	2.0552E-03	1.4687	1.2760E-02	1.1932
	400	1.1160E-04	1.6370	1.3448E-03	1.4742	9.0480E-03	1.1949
	500	7.7205E-05	1.6514	9.6711E-04	1.4776	6.9288E-03	1.1959

Numerical Examples

Table: Convergence rates with $\alpha \in (1, 2)$ for $L2_1$ discretisation

	α	1.2		1.5		1.8	
ρ	N	Error	Rate	Error	Rate	Error	Rate
	200	6.7712E-06	–	1.0410E-04	–	1.1398E-03	–
$1/2$	300	3.3365E-06	1.7455	5.6708E-05	1.4982	7.0018E-04	1.2018
	400	2.0167E-06	1.7500	3.6850E-05	1.4984	4.9559E-04	1.2013
	500	1.3638E-06	1.7531	2.6377E-05	1.4985	3.7908E-04	1.2010
	200	4.4051E-04	–	3.8011E-03	–	2.0750E-02	–
2	300	2.2173E-04	1.6931	2.0877E-03	1.4779	1.2782E-02	1.1948
	400	1.3580E-04	1.7041	1.3631E-03	1.4818	9.0608E-03	1.1962
	500	9.2697E-05	1.7113	9.7883E-04	1.4842	6.9370E-03	1.1969

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Thank you all for your attention!!!