High-order schemes based on extrapolation for semilinear fractional differential equation

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Outline

Asymptotic expansion of the error to approximate the Riemann-Liouville fractional derivative with order $\alpha \in (1, 2)$

Asymptotic expansion of the error to approximate the linear fractional differential equations

Asymptotic expansion of the error to approximate the semilinear fractional differential equations

Numerical examples

Future works

Semilinear fractional differential equation with order $\alpha \in (1,2)$

Consider, with $\alpha \in (1,2)$,

$$ {}_0^C D_t^\alpha y(t) = \beta y(t) + f(y(t)), \quad 0 \le t \le T, \tag{1} $$
$$ y(0) = y_0, \quad y'(0) = y_0^1, \tag{2} $$

- $\beta < 0, \ y_0, y_0^1 \in \mathbb{R}$,
- f satisfies the global Lipschitz condition:

$$ |f(s_1) - f(s_2)| \le L|s_1 - s_2|. $$

The equation (1)-(2) is equivalent to

$$ {}_0^R D_t^\alpha \left[y(t) - y(0) - \frac{y'(0)}{1!} t \right] + \beta y(t) = f(y(t)). $$

Here $0^C D_t^\alpha y(t)$ and $0^R D_t^\alpha y(t)$ denote the Caputo and Riemann-Liouville fractional derivatives, respectively.

Hadamard finite-part integral

Let $p \notin \mathbb{N}$ and $p > 1$, the Hadamard finite-part integral on a general interval $[a, b]$ is defined as follows (Diethelm (1997)):

$$\neq\!\!\!\!\int_a^b (x-a)^{-p} f(x) dx := \sum_{k=0}^{\lfloor p \rfloor - 1} \frac{f^{(k)}(a)(b-a)^{k+1-p}}{(k+1-p)k!} + \int_a^b (x-a)^{-p} R_{\lfloor p \rfloor - 1}(x, a) dx,$$

where $R_\mu(x, a) := \frac{1}{\mu!} \int_a^x (x-y)^\mu f^{(\mu+1)}(y) dy$ and $\neq\!\!\!\!\int$ denotes the Hadamard finite-part integral. $\lfloor p \rfloor$ denotes the largest integer not exceeding p, where $p \notin \mathbb{N}$.

For example, with $p > 1$,

$$\neq\!\!\!\!\int_a^b (x-a)^{-p} dx = \frac{1}{-p+1}(b-a)^{-p+1}.$$

Relation between Riemann-Liouville fractional derivative and Hadamard finite-part integral

It is well-known that the Riemann-Liouville fractional derivative for $\alpha \in (1,2)$ can be written as

$$
\begin{aligned}
{}_0^R D_t^\alpha f(t) &= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-s)^{1-\alpha} f(s) ds \\
&= \frac{1}{\Gamma(-\alpha)} \oint_0^t (t-s)^{-\alpha-1} f(s) ds \\
&= \frac{1}{\Gamma(-\alpha)} \oint_0^1 (tw)^{-\alpha-1} f(t-tw) t dw \\
&= \frac{t^{-\alpha}}{\Gamma(-\alpha)} \oint_0^1 w^{-\alpha-1} f(t-tw) dw,
\end{aligned}
$$

where the integral $\oint_0^1$ is interpreted as the Hadamard finite-part integral, see Diethelm (2010)

Approximate the Riemann-Liouville fractional derivative with order $\alpha \in (1,2)$

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$.

$$ {}_0^R D_t^\alpha f(t_n) = \frac{t_n^{-\alpha}}{\Gamma(-\alpha)} \oint_0^1 w^{-\alpha-1} f(t_n - t_n w) dw. $$

Denote $g(w) = f(t_n - t_n w)$. Let $w_l = \frac{l}{n}, l = 0, 1, 2, \ldots, n,\ n \geq 2$. Approximate $g(w)$ by the piecewise quadratic interpolation polynomial $g_2(w)$.

On $[w_0, w_1]$, we use $g(w_0), g(w_1), g(w_2)$,

$$ \begin{aligned} g_2(w) = & \frac{(w-w_1)(w-w_2)}{(w_0-w_1)(w_0-w_2)} g(w_0) + \frac{(w-w_0)(w-w_2)}{(w_1-w_0)(w_1-w_2)} g(w_1) \\ & + \frac{(w-w_0)(w-w_1)}{(w_2-w_0)(w_2-w_1)} g(w_2), \quad w \in [w_0, w_1], \end{aligned} $$

Approximate the Riemann-Liouville fractional derivative with order $\alpha \in (1,2)$

On $[w_{k-1}, w_k]$, $k \geq 2$, we use $g(w_{k-2}), g(w_{k-1}), g(w_k)$),

$$\begin{aligned} g_2(w) = & \frac{(w-w_{k-1})(w-w_k)}{(w_{k-2}-w_{k-1})(w_{k-2}-w_k)} g(w_{k-2}) \\ & + \frac{(w-w_{k-2})(w-w_k)}{(w_{k-1}-w_{k-2})(w_{k-1}-w_k)} g(w_{k-1}) \\ & + \frac{(w-w_{k-2})(w-w_{k-1})}{(w_k-w_{k-2})(w_k-w_{k-1})} g(w_k), \quad w \in [w_{k-1}, w_k], \end{aligned} \tag{3}$$

Approximate the Riemann-Liouville fractional derivative with order $\alpha \in (1,2)$

Lemma

Let $0 = t_0 < t_1 < \cdots < t_N = T$ *with* $N \geq 2$ *be a partition of* $[0, T]$ *and* $\tau = \frac{T}{N}$ *the step size. Let* $\alpha \in (1,2)$. *Then, with* $n \geq 2$,

$$
{}_0^R D_t^{\alpha} f(t_n) = \tau^{-\alpha} \sum_{k=0}^{n} w_{kn} f(t_{n-k}) + O(\tau^{3-\alpha}), \tag{4}
$$

where some suitable weights $w_{kn}, k = 0, 1, 2, \ldots, n$ *and* $n = 2, 3, \ldots, N$.

Other higher order numerical methods: Chen and Li (2016), Du et al (2010), Gao and Sun (2016), Hao et al (2021), Jin et al (2016), Lyu et al (2020), Qiao et al 2020 Shen et al (2020), Sun and Wu (2006), Zhao et al. (2015).

The error formula for the approximation of the Riemann-Liouville fractional derivative

Theorem

Let $\alpha \in (1,2)$ and let f be sufficiently smooth on $[0, T]$.

$$
\begin{aligned}
{}_0^R D_t^{\alpha} f(t_n) &= \frac{t_n^{-\alpha}}{\Gamma(-\alpha)} \left[\neq\!\!\!\!\!\int_0^1 w^{-\alpha-1} g_2(w) dw + R_n(g) \right] \\
&= \frac{t_n^{-\alpha}}{\Gamma(-\alpha)} \left[\sum_{k=0}^{n} \alpha_{kn} g\left(\frac{k}{n}\right) + R_n(g) \right],
\end{aligned} \tag{5}
$$

Here

$R_n(g) =$
$\left(d_3 n^{\alpha-3} + d_4 n^{\alpha-4} + d_5 n^{\alpha-5} + \dots\right) + \left(d_2^* n^{-4} + d_3^* n^{-6} + d_4^* n^{-8} + \dots\right)$,

In particular, for $n = N$, there holds for $t_n = t_N = T = 1$

$R_N(g) =$
$\left(d_3 \tau^{3-\alpha} + d_4 \tau^{4-\alpha} + d_5 \tau^{5-\alpha} + \dots\right) + \left(d_2^* \tau^4 + d_3^* \tau^6 + d_4^* \tau^8 + \dots\right)$.

The error formula for the approximation of the Riemann-Liouville fractional derivative

Proof:

Step 1.

$$
\begin{aligned}
R_n(g) &= \fint_0^1 w^{-\alpha-1} g(w)\,dw - \fint_0^1 w^{-\alpha-1} g_2(w)\,dw \\
&= \fint_{w_0}^{w_1} w^{-\alpha-1}\big(g(w) - g_2(w)\big)\,dw \\
&\quad + = \sum_{l=1}^{n-1} \fint_{w_l}^{w_{l+1}} w^{-\alpha-1}\big(g(w) - g_2(w)\big)\,dw = I_1 + I_2.
\end{aligned}
$$

For I_1, we have

$$
\begin{aligned}
I_1 = \fint_0^1 (w_0 + hs)^{-\alpha-1} \Big[g(w_0 + hs) - \Big(\frac{1}{2}(s-1)(s-2) g(w_0) \\
- s(s-2) g(w_1) + \frac{1}{2} s(s-1) g(w_2) \Big) \Big] h\,ds.
\end{aligned}
$$

The error formula for the approximation of the Riemann-Liouville fractional derivative

Proof: Since g is sufficiently smooth, by using the Taylor series expansion, we find for $k = 0, 1, 2$ that

$$\begin{aligned} g(w_k) =& g(w_0 + hs) + \frac{g^{(1)}(w_0 + hs)}{1!}(hk - hs) \\ &+ \frac{g^{(2)}(w_0 + hs)}{2!}(hk - hs)^2 + \frac{g^{(3)}(w_0 + hs)}{3!}(hk - hs)^3 + \dots . \end{aligned}$$

We get

$$\begin{aligned} I_1 =& \int_0^1 (w_0 + hs)^{-\alpha-1} \Big[h^3 g^{(3)}(w_0 + hs)\pi_0(s) + h^4 g^{(4)}(w_0 + hs)\pi_1(s) \\ &+ h^5 g^{(5)}(w_0 + hs)\pi_2(s) + \dots \Big] h \, ds \\ =& \sum_{k=0}^{+\infty} h^{k+3} \int_0^1 \Big[h(w_0 + hs)^{-\alpha-1} g^{(k+3)}(w_0 + hs) \Big] \pi_k(s) \, ds, \end{aligned}$$

The error formula for the approximation of the Riemann-Liouville fractional derivative

Proof: Note that

$$h(w_0 + hs)^{-\alpha-1} g^{(l)}(w_0 + hs) = h^{-\alpha} \sum_{k=0}^{\infty} b_{kl}(s) h^k + \sum_{k=0}^{\infty} a_{kl}(s) h^k,$$

for some suitable functions $a_{kl}(s), b_{kl}(s), k = 0, 1, \dots$ and $l = 3, 4, \dots,$ which are not necessarily the same at different occurrences. Hence, we obtain

$$I_1 = \left(d_3 h^{3-\alpha} + d_4 h^{4-\alpha} + d_5 h^{5-\alpha} + \dots\right) + \left(d_2^* h^4 + d_3^* h^6 + d_4^* h^8 + \dots\right),$$

The error formula for the approximation of the Riemann-Liouville fractional derivative

Proof: For I_2, we have

$$
\begin{aligned}
I_2 &= \sum_{l=1}^{n-1} \neq\!\!\!\!\!\!\int_0^1 (w_l + hs)^{-\alpha-1} \Big[g(w_l + hs) - g_2(w) \Big] \, dw \\
&= \sum_{l=1}^{n-1} \neq\!\!\!\!\!\!\int_0^1 (w_l + hs)^{-\alpha-1} \Big[g(w_l + hs) - \Big(\frac{1}{2}(s-1)(s-2) g(w_l) \\
&\quad - s(s-2) g(w_{l+1}) + \frac{1}{2} s(s-1) g(w_{l+2}) \Big) \Big] \, h \, ds.
\end{aligned}
$$

The error formula for the approximation of the Riemann-Liouville fractional derivative

Proof: A use of the Taylor series expansion as in the estimate of I_1 shows

$$\begin{aligned} I_2 =& h \sum_{l=1}^{n-1} \int_0^1 (w_l + hs)^{-\alpha-1} \Big[h^3 g^{(3)}(w_l + hs)\pi_0(s) + h^4 g^{(4)}(w_l + hs)\pi_1(s \\ &+ h^5 g^{(5)}(w_l + hs)\pi_2(s) + \dots \Big] \, ds \\ =& \sum_{k=0}^{\infty} h^{k+3} \int_0^1 \Big[h \sum_{l=1}^{n-1} (w_l + hs)^{-\alpha-1} g^{(k+3)}(w_l + hs) \Big] \pi_k(s) \, ds. \end{aligned}$$

Hence

$$I_2 = \big(d_3 h^{3-\alpha} + d_4 h^{4-\alpha} + d_5 h^{5-\alpha} + \dots\big) + \big(d_2^* h^4 + d_3^* h^6 + d_4^* h^6 + \dots\big).$$

Combining these estimates completes the proof. (Note that $h = \frac{1}{n}$)

Linear fractional differential equations

Consider

$$ {}_0^R D_t^{\alpha}\left[y(t) - y_0 - \frac{y'(0)}{1!}t\right]\Big|_{t=t_j} = \beta y(t_j) + f(t_j). \tag{6} $$

The exact solution is:

$$ y(t_j) = \frac{1}{\alpha_{0,j} - t_j^{\alpha}\Gamma(-\alpha)\beta}\left[t_j^{\alpha}\Gamma(-\alpha)\left(f(t_j) + \frac{t_j^{-\alpha}}{\Gamma(1-\alpha)}y(0) + \frac{t_j^{1-\alpha}}{\Gamma(2-\alpha)}y'(\right.\right. $$

$$ \left. - \sum_{k=1}^{j}\alpha_{kj}y(t_{j-k}) - R_j(g)\right]. \tag{7} $$

The approximate solution is:

$$ y_j = \frac{1}{\alpha_{0,j} - t_j^{\alpha}\Gamma(-\alpha)\beta}\left[t_j^{\alpha}\Gamma(-\alpha)\left(f(t_j) + \frac{t_j^{-\alpha}}{\Gamma(1-\alpha)}y(0) + \frac{t_j^{1-\alpha}}{\Gamma(2-\alpha)}y'(0)\right)\right. $$

$$ \left. - \sum^{j}\alpha_{kj}y_{j-k}\right], \tag{8} $$

Linear fractional differential equations

Theorem

Let $\alpha \in (1,2)$. . *Let* $y(t_j)$ *and* y_j *be the exact and the approximate solutions of* (7)*and*(8), *respectively. Assume that the function* $y \in C^{m+2}[0,1]$, $m \geq 3$. *Further assume that we obtain the exact starting values* $y_0 = y(0)$ *and* $y_1 = y(t_1)$. *Then*

$$y(t_N) - y_N = \sum_{\mu=3}^{m+1} c_\mu N^{\alpha-\mu} + \sum_{\mu=2}^{\mu^*} c_\mu^* N^{-2\mu} + \dots, \quad \text{for } N \to \infty,$$

or, since $\tau = \frac{1}{N}$,

$y(t_N) - y_N = \sum_{\mu=3}^{m+1} c_\mu \tau^{\mu-\alpha} + \sum_{\mu=2}^{\mu^*} c_\mu^* \tau^{2\mu} + \dots, \quad \text{for } \tau \to 0.$

proof: The proof is based on the technique in Diethelm and Walz (1997) where the case $\alpha \in (0,1)$ is considered.

Semilinear fractional differential equations

Consider

$$ {}_0^R D_t^{\alpha}\left[y(t) - y(0) - \frac{y'(0)}{1!}t\right] = \beta y(t) + f(y(t)), \quad 0 \leq t \leq T. \quad (9) $$

Let $y_j \approx y(t_j)$ denote the approximation of the exact solutions $y(t_j)$. Denote

$$ D_\tau^\alpha y_j := \tau^{-\alpha} \sum_{k=0}^{j} w_{kj} y_{j-k}. $$

Given the starting values y_0 and y_1, define the following numerical scheme for approximating (9)

$$ D_\tau^\alpha y_j - \frac{t_j^{-\alpha}}{\Gamma(1-\alpha)} y(0) - \frac{t_j^{1-\alpha}}{\Gamma(2-\alpha)} y'(0) = \beta y_j + f(y_j), \quad \text{for } j = 2, 3, \ldots, N \quad (10) $$

with $y_0 = y(0)$ and $y_1 = y(t_1)$.

semilinear fractional differential equations

Theorem

For $\alpha \in (1,2)$, let $y(t_j)$ and y_j be the exact and the approximate solutions of (9) and (10), respectively. Assume that the function $y \in C^{m+2}[0,1]$, $m \geq 3$. Further, assume that exact starting values $y_0 = y(0)$ and $y_1 = y(t_1)$ are known. Then, there exist coefficients $c_\mu = c_\mu(\alpha)$ and $c_\mu^* = c_\mu^*(\alpha)$ such that the error satisfies

$$y(t_N) - y_N = \sum_{\mu=3}^{m+1} c_\mu \tau^{\mu-\alpha} + \sum_{\mu=2}^{\mu^*} c_\mu^* \tau^{2\mu} + \dots, \quad \text{for } \tau \to 0. \quad (11)$$

semilinear fractional differential equations

Proof:
Step 1. Set for $j = 2, 3, \ldots, N$

$$e_j = (y(t_j) - \tilde{y}_j) + (\tilde{y}_j - y_j) =: \eta_j + \theta_j,$$

where $\tilde{y}_j,\ j = 2, 3, \ldots, N$ be the solutions of the linearized problem: with $j = 2, 3, \ldots, N$,

$$D_\tau^\alpha \tilde{y}_j - \frac{t_j^{-\alpha}}{\Gamma(1-\alpha)} y(0) - \frac{t_j^{1-\alpha}}{\Gamma(2-\alpha)} y'(0) = \beta \tilde{y}_j + f(y(t_j)), \qquad (12)$$

with $\tilde{y}_0 = y(0)$ and $\tilde{y}_1 = y(t_1)$.

semilinear fractional differential equations

Proof:
Step 2. By the error formula of the linear fractional differential equation, we have

$$\eta_j = \sum_{\mu=3}^{m+1} c_\mu \tau^{\mu-\alpha} + \sum_{\mu=2}^{\mu^*} c_\mu^* \tau^{2\mu} + \ldots, \quad \text{for } \tau \to 0. \qquad (13)$$

Step 3: Therefore, it remains to prove a similar error expansion of θ_N. Now the error equation in θ_j becomes

$$D_\tau^\alpha \theta_j = \beta \theta_j + \big(f(y(t_j)) - f(y_j) \big), \quad \text{for } j = 2, 3, \ldots, N \qquad (14)$$

with $\theta_0 = 0$ and $\theta_1 = 0$.

semilinear fractional differential equations

Proof: Step 3: With the help of

$$\delta_t \phi_{n-\frac{1}{2}} = \frac{\phi(t_n) - \phi(t_{n-1})}{\tau}, \tag{15}$$

and

$$p_l = \sum_{k=0}^{l-1} (l-k) w_{kl}, \quad l = 1, 2, \ldots, n, \ \ n = 2, 3, \ldots, N, \tag{16}$$

rewrite $D_\tau^\alpha \theta_j$ in a suitable manner in terms of $\delta_t \theta_{j-\frac{1}{2}}$ and hence, obtain an equivalent equation θ_j for $j = 1, 2, \ldots, N$ as

$$\tau^{1-\alpha} \left(p_1 \delta_t \theta_{j-\frac{1}{2}} - \sum_{l=1}^{j-1} (p_{j-l} - p_{j-l+1}) \delta_t \theta_{l-\frac{1}{2}} \right) = \beta \theta_j + \big(f(y(t_j)) - f(y_j) \big) \tag{17}$$

with $\theta_0 = 0$ and $\theta_1 = 0$.

semilinear fractional differential equations

In order to derive an estimate of θ_j, we need the following Lemma, whose proof is given in the subsection 7.2 of the Appendix.

Lemma

Let $1 < \alpha < 2$. Then, the coefficients p_l defined by (16) satisfy the following properties

$$p_l > 0, \quad l = 1, 2, \ldots, n, \;\; n = 2, 3, \ldots, N, \tag{18}$$

$$p_l > p_{l+1}, \quad l = 1, 2, \ldots, n-1, \;\; n = 2, 3, \ldots, N, \tag{19}$$

$$\sum_{l=1}^{n} p_l \leq \frac{n^{2-\alpha}}{\Gamma(3-\alpha)}, \qquad n = 2, 3, \ldots, N. \tag{20}$$

semilinear fractional differential equations

Proof: Step 3: Multiplying (17) by $\tau\delta_t\theta_{j-\frac{1}{2}}$, it follows that

$$\tau^{1-\alpha}\Big(p_1\delta_t\theta_{j-\frac{1}{2}} - \sum_{l=1}^{j-1}(p_{j-l} - p_{j-l+1})\delta_t\theta_{l-\frac{1}{2}}\Big)\Big(\tau\delta_t\theta_{j-\frac{1}{2}}\Big)$$

$$-\beta\theta_j\Big(\tau\delta_t\theta_{j-\frac{1}{2}}\Big) = \big(f(y(t_j)) - f(y_j)\big)\Big(\tau\delta_t\theta_{j-\frac{1}{2}}\Big).$$

semilinear fractional differential equations

Proof: Step 3: Note that

$$\begin{aligned}
&\Big[\sum_{l=1}^{j-1}(p_{j-l}-p_{j-l+1})\delta_t\theta_{l-\frac{1}{2}}\Big]\delta_t\theta_{j-\frac{1}{2}} \\
&\leq \sum_{l=1}^{j-1}(p_{j-l}-p_{j-l+1})\,\frac{1}{2}\Big(\big|\delta_t\theta_{l-\frac{1}{2}}\big|^2+\big|\delta_t\theta_{j-\frac{1}{2}}\big|^2\Big) \\
&=\frac{1}{2}\sum_{l=1}^{j-1}(p_{j-l}-p_{j-l+1})\big|\delta_t\theta_{l-\frac{1}{2}}\big|^2+\frac{1}{2}(p_1-p_j)\big|\delta_t\theta_{j-\frac{1}{2}}\big|^2 \\
&=\frac{1}{2}\sum_{l=1}^{j-1}p_{j-l}\big|\delta_t\theta_{l-\frac{1}{2}}\big|^2-\frac{1}{2}\sum_{l=1}^{j}p_{j-l+1}\big|\delta_t\theta_{l-\frac{1}{2}}\big|^2 \\
&+\frac{1}{2}p_1\big|\delta_t\theta_{j-\frac{1}{2}}\big|^2+\frac{1}{2}(p_1-p_j)\big|\delta_t\theta_{j-\frac{1}{2}}\big|^2,
\end{aligned}$$

semilinear fractional differential equations

Proof: Step 3: and

$$-\beta\theta_j\left(\tau\delta_t\theta_{j-\frac{1}{2}}\right) = -\beta\theta_j\left(\theta_j - \theta_{j-1}\right) \geq -\frac{\beta}{2}\left(|\theta_j|^2 - |\theta_{j-1}|^2\right),$$

and

$$\begin{aligned}&\left(f(y(t_j)) - f(y_j)\right)\left(\tau\delta_t\theta_{j-\frac{1}{2}}\right) \leq L|y(t_j) - y_j|\left|\tau\delta_t\theta_{j-\frac{1}{2}}\right| \\ &\leq \frac{1}{2}\tau^{\alpha}\frac{L^2}{p_j}\left(|\eta_j|^2 + |\theta_j|^2\right) + \frac{1}{2}p_j\tau^{2-\alpha}\left|\delta_t\theta_{j-\frac{1}{2}}\right|^2,\end{aligned} \tag{21}$$

semilinear fractional differential equations

Proof: Step 3:

$$\tau^{1-\alpha}\left(p_1\delta_t\theta_{j-\frac{1}{2}}\right) - \tau^{2-\alpha}\Big[\frac{1}{2}\sum_{l=1}^{j-1} p_{j-l}\left|\delta_t\theta_{l-\frac{1}{2}}\right|^2 - \frac{1}{2}\sum_{l=1}^{j} p_{j-l+1}\left|\delta_t\theta_{l-\frac{1}{2}}\right|^2$$

$$+\frac{1}{2}p_1\left|\delta_t\theta_{j-\frac{1}{2}}\right|^2 + \frac{1}{2}(p_1 - p_j)\left|\delta_t\theta_{j-\frac{1}{2}}\right|^2\Big] - \frac{\beta}{2}\left(|\theta_j|^2 - |\theta_{j-1}|^2\right)$$

$$\leq \frac{1}{2}\tau^{\alpha}\frac{L^2}{p_j}\left(|\eta_j|^2 + |\theta_j|^2\right) + \frac{1}{2}p_j\tau^{2-\alpha}\left|\delta_t\theta_{j-\frac{1}{2}}\right|^2.$$

semilinear fractional differential equations

Proof: Step 3: Denoting

$$E_j = -\beta|\theta_j|^2 + \tau^{2-\alpha} \sum_{l=1}^{j} p_{j-l+1} |\delta_t \theta_{l-\frac{1}{2}}|^2,$$

we obtain

$$E_j \leq E_{j-1} + \frac{C(L)}{p_j} \tau^{\alpha} \left(|\eta_j|^2 + |\theta_j|^2 \right). \tag{22}$$

It follows using $1/p_j = \Gamma(2-\alpha)\tau^{1-\alpha}\, t_j^{\alpha-1}, j \geq 2$, $\theta_0 = 0$, $\theta_1 = 0$, and $\tau \sum_{l=2}^{j} t_l^{\alpha-1} \leq C t_j^{\alpha}$ that

$$\begin{aligned} E_j &\leq E_1 + C\,\Gamma(2-\alpha)\,\tau \sum_{l=2}^{j} t_l^{\alpha-1} \left(|\eta_l|^2 + |\theta_l|^2 \right) \\ &\leq C(L,\alpha)\, t_j^{\alpha} \max_{1 \leq l \leq j} |\eta_l|^2 + C(L,\alpha)\, t_j^{\alpha-1} \tau \sum_{l=2}^{j} |\theta_l|^2. \end{aligned} \tag{23}$$

semilinear fractional differential equations

Proof: Step 3: A use of Gronwall's inequality yields

$$|\theta_j|^2 \leq C(L, T, \alpha) \max_{1 \leq l \leq j} |\eta_l|^2, \tag{24}$$

which implies that

$$|e_j| = |\eta_j| + |\theta_j| \leq C(L, T, \alpha) \max_{1 \leq l \leq j} |\eta_l|,$$

where η_l, by (13), has the asymptotic expansion. This completes the rest of the proof.

Richardson extrapolation method

Let $A_0(\tau)$ denote the approximation of A calculated by an algorithm with the step size τ. Assume that

$$A = A_0(\tau) + a_0\tau^{\lambda_0} + a_1\tau^{\lambda_1} + a_2\tau^{\lambda_2} + \dots \quad \text{as } \tau \to 0, \tag{25}$$

where $a_j, j = 0, 1, 2, \dots$ are unknown constants and $0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots$ are some positive numbers.
Denote, with $b = 2$,

$$A_1(\tau) = \frac{b^{\lambda_0} A_0\left(\frac{\tau}{b}\right) - A_0(\tau)}{b^{\lambda_0} - 1}, \tag{26}$$

we then arrive at $A = A_1(\tau) + b_1\tau^{\lambda_1} + b_2\tau^{\lambda_2} + \dots \quad \text{as } \tau \to 0,$ for some suitable constants $b_1, b_2, \dots$.
Similarly we may construct $A_2(\tau), A_3(\tau), \dots$.
Extrapolation methods for fractional differentil equations: Hao et al. (2021), Li et al. (2022), Qi and Sun (2022),

Numerical example 1

Denote $A = {}_0^R D_t^{\alpha} f(t_N)$ and approximate A by $A_0(\tau) = \tau^{-\alpha} \sum_{k=0}^{N} w_{kN} f(t_{N-k})$. Then we have

$$A = A_0(\tau) + \left(d_3\tau^{3-\alpha} + d_4\tau^{4-\alpha} + d_5\tau^{5-\alpha} + \dots\right) + \left(d_2^*\tau^4 + d_3^*\tau^6 + d_4^*\tau^8 + \dots\right).$$

In Table 1, we choose $f(t) = t^5, \tau = 1/20$, $b = 2$ and $T = 1$. We obtain the approximate solutions with the step sizes $\left(\tau, \frac{\tau}{2}, \frac{\tau}{2^2}, \frac{\tau}{2^3}, \frac{\tau}{2^4}, \frac{\tau}{2^5}\right) = \left(\frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}, \frac{1}{640}\right)$.

Numerical example 1

Step size	Error of the scheme (4)	Extrapolated values	
1/20	4.17e-01		
1/40	1.53e-01	8.34e-03	
1/80	5.51e-02	1.51e-03	4.18e-05
1/160	1.96e-02	2.70e-04	4.26e-06
1/320	6.97e-03	4.82e-05	4.47e-07
1/640	2.47e-03	8.56e-06	4.85e-08
EOC	1.48 (1.50)	2.48 (2.50)	3.25 (3.50)

Table: Errors for approximating ${}_0^R D_t^{\alpha}(t^5)$ with $\alpha = 1.5$, taken at $T = 1$

Numerical example 1

Step size	Error of the scheme (4)	Extrapolated values	
1/20	1.96e-02		
1/40	7.06e-03	1.91e-04	
1/80	2,52e-03	3.43e-05	8.73e-07
1/160	8.94e-04	6.14e-06	8.86e-08
1/320	3.17e-04	1.09e-06	9.23e-09
1/640	1.12e-04	1.94e-07	9.81e-10
EOC	1.49 (1.50)	2.48 (2.50)	3.27 (3.50)

Table: Errors for approximating ${}_0^R D_t^{\alpha}\big(\cos \pi t\big)$ with $\alpha = 1.5$, taken at $T = 1$

Numerical example 1

Step size	Error of the scheme (4)	Extrapolated values	
$1/20$	5.33e-02		
$1/40$	1.92e-02	5.16e-04	
$1/80$	6.84e-03	9.32e-05	2.41e-06
$1/160$	2.43e-03	1.67e-05	2.46e-07
$1/320$	8.61e-04	2.97e-06	2.58e-08
$1/640$	3.05e-04	5.27e-07	2.74e-09
EOC	1.49 (1.50)	2.48 (2.50)	3.26 (3.50)

Table: Errors for approximating ${}_0^R D_t^{\alpha}\left(e^t\right)$ with $\alpha = 1.5$, taken at $T = 1$

Numerical example 2

Consider the following linear fractional differential equation

$$ {}_0^R D_t^\alpha \left[y(t) - y(0) - \frac{y'(0)}{1!} t \right] = \beta y(t) + f(t), \quad 0 \le t \le T, \quad (27) $$

where $y(t) = t^5$ and $\beta = -1$ and $f(t) = {}_0^R D_t^\alpha t^5 + t^5$. The initial values are $y_0 = y_0^1 = 0$.

Let $A = y(t_N)$ with $T_N = 1$ be the exact solution of (27). Let $A_0(\tau) = y_N$ be the approximate solution obtained from (8). By Theorem 4, we arrive at

$$ y(t_N) - y_N = \left(c_3 \tau^{3-\alpha} + c_4 \tau^{4-\alpha} + c_5 \tau^{5-\alpha} + \dots \right) + \left(c_2^* \tau^4 + c_3^* \tau^6 + c_4^* \tau^8 \right. \quad (28) $$

In Tables 4, 5, we choose $\tau = 1/20$, $b = 2$, $y_0 = 0$ and $y_1 = \tau^5$. We obtain the extrapolated values of the approximate solutions with the step sizes $\left(\tau, \frac{\tau}{2}, \frac{\tau}{2^2}, \frac{\tau}{2^3}, \frac{\tau}{2^4}, \frac{\tau}{2^5}\right) = \left(\frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}, \frac{1}{640}\right)$.

Numerical example 2

Step size	Scheme (8)	Extrapolated values		S
1/20	2.69e-02			
1/40	7.85e-03	1.78e-04		
1/80	2.27e-03	2.76e-05	2.41e-06	
1/160	6.56e-04	4.17e-06	2.32e-07	
1/320	1.88e-04	6.14e-07	1.80e-08	
1/640	5.43e-05	8.94e-08	1.47e-09	
EOC	1.79 (1.80)	2.74 (2.80)	3.56 (3.80)	
CPU times	0.2576 seconds	0.0012 seconds	0.0012 seconds	

Table: Errors for equation (27) with $\alpha = 1.2$, taken at $T = 1$

Numerical example 2

Step size	Scheme (8)	Extrapolated values		S
1/20	1.59e-01			
1/40	7.24e-02	5.80e-03		
1/80	3.23e-02	1.32e-03	7.40e-05	
1/160	1.42e-02	3.07e-04	2.49e-05	
1/320	6.23e-03	7.10e-05	5.40e-06	
1/640	2.72e-03	1.63e-05	1.06e-06	
EOC	1.19 (1.20)	2.13 (2.20)	2.90 (3.20)	
CPU times	0.2584 seconds	0.0012 seconds	0.0012 seconds	

Table: Errors for equation (27) with $\alpha = 1.8$, taken at $T = 1$

Numerical example 3

Consider the following semilinear fractional differential equation

$$ {}_0^R D_t^{\alpha}\left[y(t) - y(0) - \frac{y'(0)}{1!}t\right] = \beta y(t) + f(y(t)) + g(t), \quad 0 \leq t \leq T, \tag{29} $$

where $y(t) = t^5$, $\beta = -1$, $f(y) = \sin(y)$ and
$g(t) = {}_0^R D_t^{\alpha} t^5 + t^5 - \sin(t^5)$.
For given $y_0 = y(0) = 0$, $y_1 = y(\tau) = \tau^5$, we define the following numerical method, with $n \geq 2$,

$$ \begin{aligned} w_0 y_n - \tau^{\alpha}\beta y_n - \tau^{\alpha} f(y_n) &= -\sum_{j=1}^{n} w_j y_{n-j} + \tau^{\alpha} g(t_n) \\ &+ \tau^{\alpha}\left(\frac{\Gamma(1)}{\Gamma(1-\alpha)} t_n^{-\alpha}\right) y(0) + \tau^{\alpha}\left(\frac{\Gamma(2)}{\Gamma(2-\alpha)} t_n^{1-\alpha}\right) y'(0). \end{aligned} \tag{30} $$

Numerical example 3

Let $A = y(t_N)$ with $T_N = 1$ be the exact solution of (29). Let $A_0(\tau) = y_N$ be the approximate solution obtained from (30) by using MATLAB function "fsolve.m".

In Table 6, we choose $\tau = 1/20$, $b = 2$. We obtain the extrapolated values of the approximate solutions with the step sizes $\left(\tau, \frac{\tau}{2}, \frac{\tau}{2^2}, \frac{\tau}{2^3}, \frac{\tau}{2^4}, \frac{\tau}{2^5}\right) = \left(\frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}, \frac{1}{640}\right)$.

Numerical example 3

Step size	Error of the scheme (30)	Extrapolated values	
1/20	1.89e-01		
1/40	7.03e-02	5.04e-03	
1/80	2.57e-02	1.40e-03	6.24e-04
1/160	9.32e-03	3.23e-04	9.06e-05
1/320	3.35e-03	8.20e-05	3.02e-05
1/640	1.19e-03	9.52e-06	-6.03e-06
EOC	1.46 (1.50)	2.26 (2.50)	3.15 (3.50)

Table: Errors for equation (29) with $\alpha = 1.5$, taken at $T = 1$

Numerical example 4

Consider the following semilinear fractional differential equation

$$ {}_0^RD_t^\alpha\left[y(t)-y(0)-\frac{y'(0)}{1!}t\right]=\beta y(t)+f(y(t))+g(t),\quad 0\leq t\leq T, \tag{31} $$

where $y(t)=t^5$, $\beta=-1$, $f(y)=y-y^3$ and
$g(t)={}_0^RD_t^\alpha t^5+t^5-\left(t^5-(t^5)^3\right)$.

Numerical example 4

Step size	Error of the scheme (30)	Extrapolated values	
1/20	1.71e-01		
1/40	6..59e-02	8.44e-03	
1/80	2.44e-02	1.79e-03	3.71e-04
1/160	8.89e-03	3.70e-04	6.35e-05
1/320	3.20e-03	8.68e-05	2.59e-05
1/640	1.41e-03	1.54e-05	1.36e-07
EOC	1.44 (1.50)	2.27 (2.50)	3.40 (3.50)

Table: Errors for equation (29) with $\alpha = 1.5$, taken at $T = 1$

Numerical example 5

Consider the following linear fractional differential equation

$$ {}_0^R D_t^\alpha \left[y(t) - y(0) - \frac{y'(0)}{1!} t \right] = \beta y(t) + f(t), \quad 0 \le t \le T, \quad (32) $$

where $y(t) = t^\gamma, \gamma > 1$ and $\beta = -1$ and $f(t) = {}_0^R D_t^\alpha t^\gamma + t^\gamma$. The initial values are $y_0 = y_0^1 = 0$.

In this example, we consider the case where the solution is not sufficiently smooth. We shall choose $\gamma = 1.4$ and the exact solution takes $y(t) = t^{1.4}$ which is not sufficiently smooth. In Table 8 we choose $\alpha = 1.4$, $\tau = 1/20$, $b = 2$, $y_0 = 0$ and $y_1 = \tau^{1.4}$. We obtain the extrapolated values of the approximate solutions with the step sizes $\left(\tau, \frac{\tau}{2}, \frac{\tau}{2^2}, \frac{\tau}{2^3}, \frac{\tau}{2^4}, \frac{\tau}{2^5}\right) = \left(\frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}, \frac{1}{640}\right)$.

Numerical example 5

Step size	Scheme (8)	Extrapolated values	
1/20	1.26e-02		
1/40	5.03e-03	1.30e-03	
1/80	2.11e-03	6.69e-04	5.45e-04
1/160	9.35e-04	3.57e-04	2.95e-04
1/320	4.32e-04	1.85e-04	1.51e-04
1/640	2.06e-04	9.44e-05	7.64e-05
EOC	1.18 (1.80)	0.94 (2.80)	0.94 (3.80)

Table: Errors for equation (32) with $\alpha = 1.4$, taken at $T = 1$ in Example **??**

Future works

- Consider the extrapolation nmethod for nonsmooth solutions
- Consider the extrapolation method for nonlinear time fractional wave equation.
- Consider the extrapolation method for nonlinear space fractional partial differential equations.

THANK YOU VERY MUCH FOR YOUR ATTENSION!

ANY QUESTIONS