Uniqueness in the inverse scattering problem in a piecewise homogeneous medium

To cite this article: Xiaodong Liu et al 2010 Inverse Problems 26 015002

View the article online for updates and enhancements.

Related content
- The inverse electromagnetic scattering problem in a piecewise homogeneous medium
  Xiaodong Liu, Bo Zhang and Jiaqing Yang
- A uniqueness result for the inverse electromagnetic scattering problem
  Xiaodong Liu and Bo Zhang
- Topical Review
  David Colton and Rainer Kress

Recent citations
- A novel sampling method for multiple multiscale targets from scattering amplitudes at a fixed frequency
  Xiaodong Liu
- Recovery of an embedded obstacle and its surrounding medium from formally determined scattering data
  Hongyu Liu and Xiaodong Liu
- A modification of the factorization method for the classical acoustic inverse scattering problems
  Andreas Kirsch and Xiaodong Liu
Uniqueness in the inverse scattering problem in a piecewise homogeneous medium

Xiaodong Liu, Bo Zhang and Guanghui Hu

LSEC and Institute of Applied Mathematics, AMSS, Chinese Academy of Sciences,
Beijing 100190, People’s Republic of China

E-mail: lxd230@163.com, b.zhang@amt.ac.cn and huguanghui@amss.ac.cn

Received 20 May 2009, in final form 26 August 2009
Published 11 December 2009
Online at stacks.iop.org/IP/26/015002

Abstract
The scattering of time-harmonic acoustic plane waves by an impenetrable obstacle in a piecewise homogeneous medium is considered. Having established the well posedness of the direct problem by the variational method, we prove a uniqueness result for the inverse problem, that is, the unique determination of the obstacle and its boundary condition from a knowledge of the far-field pattern for incident plane waves. The proof is based on a generalization of the mixed reciprocity relation.

1. Introduction

In this paper, we consider the problem of scattering of time-harmonic acoustic plane waves by an impenetrable obstacle surrounded by a piecewise homogeneous medium. In practical applications, the background might not be homogeneous and then must be modeled as a layered medium. A medium of this type that is a nested body consisting of a finite number of homogeneous layers occurs in various areas of applications such as radar, remote sensing, geophysics and nondestructive testing.

Let \( \Omega \) denote the piecewise homogeneous medium which is a bounded and closed subset of \( \mathbb{R}^n \) \((n = 2, 3)\) with a \(C^2\) boundary \( \partial \Omega \). Let \( \Omega_0 \) be the exterior region of \( \Omega \), that is, \( \Omega_0 = \mathbb{R}^n \setminus \overline{\Omega} \) \((n = 2, 3)\). The interior of \( \Omega \) is divided by means of closed and nonintersecting \(C^2\) surfaces \( S_j \) \((j = 1, 2, \ldots, N)\) into subsets (layers) \( \Omega_j \) \((j = 1, 2, \ldots, N)\) with \( \partial \Omega_{j-1} \cap \partial \Omega_j = S_{j-1} \) \((j = 1, 2, \ldots, N + 1)\). The regions \( \Omega_j \) \((j = 0, 1, \ldots, N)\) are homogeneous media. The region \( \Omega_{N+1} \) is the impenetrable obstacle.

We now give a brief description of the direct and inverse scattering problem.

The propagation of time-harmonic acoustic waves in a piecewise homogeneous isotropic medium in \( \mathbb{R}^n \) \((n = 2, 3)\) is modeled by the reduced wave equation or Helmholtz equation with boundary conditions on their interfaces:

\[
\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega_j, \quad j = 0, 1, \ldots, N, \tag{1.1}
\]
\[ u_+ = u_- = \frac{\partial u_+}{\partial \nu} = \lambda_j \frac{\partial u_-}{\partial \nu} \quad \text{on} \quad S_j, \quad j = 0, 1, \ldots, N - 1, \quad (1.2) \]

where \( \nu \) is the unit outward normal to the boundary \( S_j \). \( u_+ \), \( \frac{\partial u_+}{\partial \nu} \) denote the limit of \( u \), \( \frac{\partial u}{\partial \nu} \) on the surface \( S_j \) from the exterior (interior) of \( S_j \) and \( \lambda_j \) represents the nonnegative constant across the surface \( S_j \) (\( j = 0, 1, \ldots, N - 1 \)). Here, \( u \) denotes the complex-valued space-dependent part of the time-harmonic acoustic wave \( u(x) e^{-i\omega t} \) and \( k_j \) is the positive wave number given by \( k_j = \omega_j / c_j \) in terms of the frequency \( \omega_j \) and the sound speed \( c_j \) in the corresponding region \( \Omega_j \) (\( j = 0, 1, \ldots, N \)). The distinct wave numbers \( k_j \) (\( j = 0, 1, \ldots, N \)) correspond to the fact that the background medium consists of several physically different materials. On these surfaces \( S_j \) (\( j = 0, 1, \ldots, N - 1 \)), the so-called transmission conditions (1.2) are imposed, which represent the continuity of the medium and equilibrium of the forces acting on it. On the boundary \( S_N \) of the obstacle \( \Omega_{N+1} \) the total wave \( u \) has to satisfy a boundary condition of the form

\[ B(u) = 0 \quad \text{on} \quad S_N. \quad (1.3) \]

For a *sound-soft* obstacle the pressure of the total wave vanishes on the boundary, so a Dirichlet boundary condition

\[ B(u) := u \quad \text{on} \quad S_N \]

is imposed. Similarly, the scattering from a *sound-hard* obstacle leads to a Neumann boundary condition

\[ B(u) := \frac{\partial u}{\partial \nu} \quad \text{on} \quad S_N \]

since the normal velocity of the total acoustic wave vanishes on the boundary. More general and realistic boundary conditions are to allow that the normal velocity on the boundary is proportional to the excess pressure on the boundary, which leads to an impedance boundary condition of the form

\[ B(u) := \frac{\partial u}{\partial \nu} + i\lambda u \quad \text{on} \quad S_N \]

with a nonnegative continuous function \( \lambda \). Henceforth, we shall use \( B(u) = 0 \) to represent either of the above three types or mixed type of boundary conditions on \( S_N \). In the region \( \mathbb{R}^n \setminus \Omega_{N+1} \) (\( n = 2, 3 \)), the total wave \( u \) is the superposition of the given incident plane wave \( u^i(x) = e^{ik_0 x \cdot d} \) and the scattered wave \( u^s(x) \) which is required to satisfy the Sommerfeld radiation condition

\[ \lim_{r \to \infty} r^{\frac{n-1}{2}} \left( \frac{\partial u^s}{\partial r} - ik_0 u^s \right) = 0 \quad (1.4) \]

uniformly in all directions \( x / |x| \), where \( r = |x| \). It physically implies that energy is transported to infinity and it is an important ingredient in ensuring that the physically correct solution of the scattering problem is selected. The well posedness (existence, uniqueness and stability) of the direct problem for a *sound-soft* obstacle using the theory of generalized solutions has been studied by Athanasiadis and Stratis [2]. However, it is not suitable for our later use, that is, the proof of the uniqueness result in the inverse problem. Therefore, following Cakoni and Colton [3] and Mclean [19], we will give a new proof and consider a general mixed boundary value problem in section 2. Moreover, it is known that \( u^s(x) \) has the following asymptotic
Inverse Problems 26 (2010) 015002 X Liu et al

representation:

\[
\phi(x, d) = \frac{e^{ik_0 |x|}}{|x|^{n-1}} \left\{ u^\infty(\hat{x}, d) + O\left(\frac{1}{|x|}\right)\right\} \quad \text{as} \quad |x| \to \infty \quad (1.5)
\]

uniformly for all directions \( \hat{x} := x/|x| \), where the function \( u^\infty(\hat{x}, d) \) defined on the unit sphere \( S \) is known as the far-field pattern with \( \hat{x} \) and \( d \) denoting, respectively, the observation direction and the incident direction. By analyticity, the far-field pattern is completely determined on the whole unit sphere \( S \) by only knowing it on some open subset \( S^* \) of \( S \) \cite{8}. Therefore, all the uniqueness results carry over to the case of limited aperture problems where the far-field pattern is only known on some open subset \( S^* \) of \( S \). Without loss of generality, we can assume that the far-field data are given on the whole unit sphere \( S \), that is, in every possible observation direction.

The inverse problem we consider in this paper is, given the wave numbers \( k_j \) (\( j = 0, 1, \ldots, N \)), the nonnegative constants \( \lambda_j \) (\( j = 0, 1, \ldots, N - 1 \)) and the far-field pattern \( u^\infty(\hat{x}, d) \) for all incident plane waves with incident direction \( d \in S \), to determine the location and shape of the obstacle \( \Omega_{N+1} \) and its boundary condition. As usual in most of the inverse problems, the first question to ask in this context is the identifiability, that is, whether an obstacle can be identified from a knowledge of the far-field pattern. Mathematically, the identifiability is the uniqueness issue which is of theoretical interest and is required in order to proceed to efficient numerical methods of solutions.

In the last 30 years, both the inverse scattering problem in a homogeneous medium (i.e. \( N = 0 \)) and the inverse medium problem have obtained great development in the theoretical and numerical aspects. We refer to the monographs \cite{8, 14, 23} and the references therein for a comprehensive discussion. As far as we know, there are few uniqueness results for inverse obstacle scattering in a piecewise homogeneous medium. When the obstacle is penetrable (with transmission boundary conditions), Athanasiadis, Ramm and Stratis \cite{1} proved that the obstacle is determined uniquely by the corresponding far-field pattern based on an orthogonality result \cite{23}. Yan and Pang \cite{26} gave a proof of uniqueness of the sound-soft obstacle based on Schiffer’s idea. But their method cannot be extended to other boundary conditions. They also gave a result for the case of a sound-hard obstacle in a two-layered background medium in \cite{22} using a generalization of Schiffer’s method. However, their method is hard to be extended to the case of a multilayered background medium and seems unreasonable to require the interior wave number to be in an interval.

There are few results on uniqueness in determining an obstacle or a medium buried in an inhomogeneous medium. In 1998, Kirsch and Päiväranta \cite{15} proved that a sound-soft obstacle or a penetrable inhomogeneous medium can be uniquely determined if the outside inhomogeneity is known in advance. In the same year, Hähner \cite{13} showed that both the sound-soft obstacle and the outside inhomogeneous medium in 2D can be uniquely determined by the far-field patterns corresponding to all incident plane waves with an interval of wave numbers. Recently in \cite{20}, the authors showed that an obstacle buried in a known inhomogeneous medium can be determined from measurements of the far field at a fixed wave number without a priori knowledge of the boundary condition.

In recent years a new version of the linear sampling method based on the reciprocity gap functional has been developed for the numerical recovery of the shape of an obstacle or an inhomogeneous medium immersed in a two-layered background medium in the case when the nonnegative constants \( \lambda_0 = 1 \) (see, e.g. \cite{4–7, 10, 11} and the references therein). In particular, in \cite{10}, Cristo and Sun also proved that the obstacle and the surface impedance can be uniquely determined by the near field on \( \partial \Omega \) for all point sources on the boundary of a box containing \( \Omega \). It should be pointed out that, recently in \cite{24}, Yaman presented a numerical method based on
Newton iterations and integral equations to reconstruct the location and shape of a sound-soft obstacle buried in a two-layered background medium in 2D using the far-field patterns.

Our contribution in this paper is to provide a uniqueness result for the inverse obstacle scattering problem in a known layered background medium with arbitrary nonnegative constants $\lambda_j$ using the far-field patterns corresponding to incident plane waves.

This paper is organized as follows. In the next section, we will establish the well posedness of the direct scattering problem by the variational method. Section 3 is devoted to the unique determination of the obstacle and its boundary condition from a knowledge of the far-field pattern for incident plane waves based on a generalization of the mixed reciprocity relation. We will not assume that we know the boundary condition for the obstacle. This seems to be appropriate for a number of applications where the physical nature of the obstacle is unknown.

For simplicity, and without loss of generality, in this paper we only consider the case $N = 1$, that is, the case where the obstacle is buried in a two-layered background medium, as shown in figure 1. In this case, $\Omega_2$ is the impenetrable obstacle (see figure 1). The results obtained in this paper are also available for the case of general $N$ and can be proved similarly.

2. The direct scattering problem

We only consider the three-dimensional case. We remark that all the results of this section remain valid in two dimensions after appropriate modifications of the fundamental solution, the radiation condition and the spherical wavefunctions. The scattering of an incident field in a two-layered background medium is depicted in figure 1.

We will focus on the general case where mixed boundary conditions are imposed on the boundary $S_1$ of the obstacle $\Omega_2$. More precisely, the boundary $S_1$ consists of two parts, that is, $S_1 = S_{1,D} \cup S_{1,I}$, where $S_{1,D}$ and $S_{1,I}$ are two disjoint, relatively open subsets (possibly disconnected) of $S_1$.

Following Cakoni and Colton [3] and Mclean [19], we shall use the variational method to find a solution of the problem (1.1)–(1.4). To this end, let $D$ denote a bounded domain and let $B_R := \{ x : |x| < R \}$. Define the Sobolev spaces

$$H^1(D) := \{ u : u \in L^2(D), \nabla u \in L^2(D) \},$$

$$H^1_{\text{loc}}(\mathbb{R}^3 \setminus D) := \{ u : u \in H^1(B_R \setminus D) \text{ for every } R > 0 \text{ such that } B_R \setminus D \neq \emptyset \}.$$

Recall that $H^\frac{1}{2}(\partial D)$ is the trace space of $H^1(D)$ and $H^{-\frac{1}{2}}(\partial D)$ is the dual space of $H^\frac{1}{2}(\partial D)$. 

Figure 1. Scattering in a two-layered background medium.
In order to study the mixed boundary value problem, we need the following Sobolev spaces on an open part of the boundary. We refer the reader to [19] for a systematic treatment.

Let $\Gamma$ be an open subset of the boundary $\partial D$. Define

$$H^\frac{1}{2}(\Gamma) := \{ u|_\Gamma : u \in H^\frac{1}{2}(\partial D) \}, \quad \tilde{H}^\frac{1}{2}(\Gamma) := \{ u \in H^\frac{1}{2}(\partial D) : \text{supp}(u) \subseteq \overline{\Gamma} \}.$$ 

Both $H^\frac{1}{2}(\Gamma)$ and $\tilde{H}^\frac{1}{2}(\Gamma)$ are Hilbert spaces equipped with the restriction of the inner product of $H^\frac{1}{2}(\partial D)$. Hence, we can define the corresponding dual spaces

$$H^{-\frac{1}{2}}(\Gamma) := (\tilde{H}^\frac{1}{2}(\Gamma))^\prime = \text{the dual space of } \tilde{H}^\frac{1}{2}(\Gamma),$$

$$\tilde{H}^{-\frac{1}{2}}(\Gamma) := (H^\frac{1}{2}(\Gamma))^\prime = \text{the dual space of } H^\frac{1}{2}(\Gamma).$$

It can be shown (cf theorem A4 in [19]) that there exists a bounded extension operator $\tau : H^\frac{1}{2}(\Gamma) \rightarrow H^\frac{1}{2}(\partial D)$. An important property of $\tilde{H}^\frac{1}{2}(\Gamma)$ is that the extension by zero to the whole boundary $\partial D$ of $u \in \tilde{H}^\frac{1}{2}(\Gamma)$ is in $H^\frac{1}{2}(\partial D)$ and the extension operator is bounded from $\tilde{H}^\frac{1}{2}(\Gamma)$ to $H^\frac{1}{2}(\partial D)$. Based on these results, we can identify the dual spaces as follows:

$$H^{-\frac{1}{2}}(\Gamma) := \{ u|_\Gamma : u \in H^{-\frac{1}{2}}(\partial D) \}, \quad \tilde{H}^{-\frac{1}{2}}(\Gamma) := \{ u \in H^{-\frac{1}{2}}(\partial D) : \text{supp}(u) \subseteq \overline{\Gamma} \}.$$ 

Thus, the duality pairing can be explained as

$$\langle \cdot, \cdot \rangle_{\tilde{H}^\frac{1}{2}(\Gamma)}, \tau \rangle_{H^\frac{1}{2}(\Gamma)}, \tau \rangle_{\tilde{H}^{-\frac{1}{2}}(\Gamma)} = \langle \cdot, \cdot \rangle_{\tilde{H}^\frac{1}{2}(\partial D)}, \tau \rangle_{\tilde{H}^{-\frac{1}{2}}(\partial D)},$$

where $\tilde{v} \in H^\frac{1}{2}(\partial D)$ is the extension by zero of $u \in \tilde{H}^\frac{1}{2}(\Gamma)$ and $\tilde{v} \in H^{-\frac{1}{2}}(\partial D)$ is the extension by zero of $v \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$. 

Consider the mixed boundary value problem: given $h \in L^2(\Omega_1), \ g \in H^{-\frac{1}{2}}(S_0), \ f \in H^\frac{1}{2}(S_{1,D})$ and $p \in H^{-\frac{1}{2}}(S_{1,I})$, find $u \in H^\frac{1}{2}(\Omega_1) \cap H^1_{loc}(\Omega_0)$ such that

$$\Delta u + k_0^2 u = 0 \quad \text{in } \Omega_0, \quad \Delta u + k_0^2 u = h \quad \text{in } \Omega_1, \label{eq:1}$$

$$u^+ = u^-, \quad \frac{\partial u^+}{\partial v} - \lambda_0 \frac{\partial u^-}{\partial v} = g \quad \text{on } S_0, \label{eq:2}$$

$$u = f \quad \text{on } S_{1,D}, \label{eq:3}$$

$$\frac{\partial u}{\partial v} + i\lambda u = p \quad \text{on } S_{1,I}, \label{eq:4}$$

$$\lim_{r \to \infty} r \left( \frac{\partial u}{\partial r} - ik_0 u \right) = 0 \quad \text{on } S_{1,I}. \label{eq:5}$$

where $r = |x|, \ k_j (j = 0, 1)$ are positive wave numbers, $\lambda_0$ is a nonnegative constant and $\lambda$ is a nonnegative continuous impedance function. Here, equations (2.1) and (2.2) are understood in a distributional sense and the boundary conditions (2.3)–(2.5) are understood in the trace sense.

Remark 2.1. The case $S_{1,I} = \emptyset$ corresponds to a sound-soft obstacle, and the case $S_{1,D} = \emptyset, \lambda = 0$ corresponds to a sound-hard obstacle.

Remark 2.2. The acoustic scattering of the incident plane wave $u^i = e^{ik_0 x \cdot d}$ is a particular case of the problem (2.1)–(2.6). In particular, the scattered field $u^s$ satisfies the problem (2.1)–(2.6) with $u = u^s, \ h = (k_0^2 - k_1^2) u^i|_{\Omega_0}, \ g = (\lambda_0 - 1) \frac{\partial u^i}{\partial v}|_{\delta_0}, \ f = -u^i|_{S_{1,D}}$ and $p = \left( -\frac{\partial u^i}{\partial v} - i\lambda u^i \right)|_{S_{1,I}}$.
**Theorem 2.3.** The boundary value problem (2.1)–(2.6) admits at most one solution.

**Proof.** Clearly, it is enough to show that \( u \) vanishes identically for the homogeneous boundary value problem (2.1)–(2.6), that is, \( u = 0 \) if \( f = g = h = p = 0 \). Choose a large ball \( B_R \) centered at the origin such that \( \Omega \subset B_R \). Applying Green’s first theorem over \( B_R \setminus \Omega \), we obtain that

\[
\int_{\partial B_R} \frac{\partial \overline{u}}{\partial \nu} \, ds = \int_{B_R \setminus \Omega} (\mu \Delta \overline{u} + |\nabla \overline{u}|^2) \, dx + \int_{\partial \Omega} \frac{\partial \overline{u}}{\partial \nu} \, ds.
\]

Using Green’s first theorem over \( \Omega_1 \) again and taking into account the transmission conditions (2.3) and the boundary conditions (2.4) and (2.5), we have

\[
\int_{\partial B_R} \frac{\partial \overline{u}}{\partial \nu} \, ds = \int_{B_R \setminus \Omega} (\mu \Delta \overline{u} + |\nabla \overline{u}|^2) \, dx + \lambda_0 \int_{\Omega_1} (\mu \Delta \overline{u} + |\nabla \overline{u}|^2) \, dx + i \lambda \int_{S_1} |u|^2 \, ds. \tag{2.7}
\]

Using equations (2.1) and (2.2) and taking the imaginary part of (2.7) we have, on noting that \( k_0^2, \lambda_0, \lambda \) are nonnegative real numbers and \( \lambda \) is a nonnegative continuous function, that

\[
\Re \int_{\partial B_R} \frac{\partial \overline{u}}{\partial \nu} \, ds = \lambda_0 \int_{S_1} |u|^2 \, ds \geq 0.
\]

Thus, by Rellich’s lemma [8], it follows that \( u = 0 \) in \( \mathbb{R}^3 \setminus B_R \). By the unique continuation principle, we have \( u = 0 \) in \( \Omega_0 \). Holmgren’s uniqueness theorem [16] implies that \( u = 0 \) in \( \mathbb{R}^3 \setminus \Omega_2 \), which completes the proof of the theorem. \( \square \)

The boundary value problems arising in scattering theory are formulated in unbounded domains. In order to solve such problems by the variational method, we need to write them as an equivalent problem in a bounded domain. Choose a ball \( B_R \) centered at the origin large enough such that the domain \( \Omega \) is contained in the ball and define the Dirichlet to Neumann operator

\[
T : u \rightarrow \frac{\partial \tilde{w}}{\partial \nu} \quad \text{on} \quad \partial B_R
\]

which maps \( u \) to \( \frac{\partial \tilde{w}}{\partial \nu} \) where \( \tilde{w} \) solves the exterior Dirichlet problem for the Helmholtz equation \( \Delta \tilde{w} + k_0^2 \tilde{w} = 0 \) in \( \mathbb{R}^3 \setminus B_R \) with the Dirichlet boundary data \( \tilde{u}|_{\partial B_R} = u \). Since \( B_R \) is a ball, then, by separating variables, we can find a solution to the exterior Dirichlet problem outside \( B_R \) in the form of a series expansion involving Hankel functions. Based on this result the following important properties of the Dirichlet to Neumann operator can be established (see [8, p 116–117] or [3, theorem 5.20] for details).

**Lemma 2.4.** The Dirichlet to Neumann operator \( T \) is a bounded linear operator from \( H^{\frac{1}{2}}(\partial B_R) \) to \( H^{-\frac{1}{2}}(\partial B_R) \). Furthermore, there exists a bounded operator \( T_0 : H^{\frac{1}{2}}(\partial B_R) \rightarrow H^{-\frac{1}{2}}(\partial B_R) \) satisfying that

\[
-\int_{\partial B_R} T_0 w \overline{w} \, ds \geq C \| w \|^2_{H^{\frac{1}{2}}(\partial B_R)}
\]

for some constant \( C > 0 \) such that \( T - T_0 : H^{\frac{1}{2}}(\partial B_R) \rightarrow H^{-\frac{1}{2}}(\partial B_R) \) is compact.

We now reformulate the problem (2.1)–(2.6) as follows: given \( h \in L^2(\Omega_1), g \in H^{-\frac{1}{2}}(S_0), f \in H^{\frac{1}{2}}(S_1, u) \) and \( p \in H^{-\frac{1}{2}}(S_1, i) \), find \( u \in H^1(\Omega_1) \cap H^1(B_R \setminus \Omega) \) satisfying (2.1)–(2.5) and the equation

\[
\frac{\partial u}{\partial \nu} = Tu \quad \text{on} \quad \partial B_R. \tag{2.8}
\]
In exactly the same way as in the proof of lemma 5.22 in [3] one can show that a solution $u$ to the problem (2.1)–(2.5) and (2.8) can be extended to a solution to the scattering problem (2.1)–(2.6) and conversely, for a solution $u$ to the scattering problem (2.1)–(2.6), $u$, restricted to $BR \setminus \Omega_2$, solves the problem (2.1)–(2.5) and (2.8). Therefore, by theorem 2.3, the problem (2.1)–(2.5) and (2.8) has at most one solution. We now have the following result on the well posedness of the problem (2.1)–(2.5) and (2.8).

**Theorem 2.5.** Let $h \in L^2(\Omega_1)$, $g \in H^{-\frac{1}{2}}(S_0)$, $f \in H^\frac{1}{2}(S_1,D)$ and $p \in H^{-\frac{1}{2}}(S_{1,1})$. Then the problem (2.1)–(2.5) and (2.8) has a unique solution $u \in H^1(\Omega_1) \cap H^\frac{1}{2}(BR \setminus \Omega)$ satisfying that 
\[
\|u\|_{H^1(\Omega_1)} + \|u\|_{H^\frac{1}{2}(BR \setminus \Omega)} \leq C (\|f\|_{H^\frac{1}{2}(S_1,D)} + \|p\|_{H^{-\frac{1}{2}}(S_{1,1})} + \|h\|_{L^2(\Omega_1)} + \|g\|_{H^{-\frac{1}{2}}(S_0)})
\]
with a positive constant $C$ independent of $h, g, p$ and $f$.

**Proof.** Let $\tilde{f} \in H^\frac{1}{2}(S_1)$ be the extension of the Dirichlet data $f \in H^\frac{1}{2}(S_1,D)$ satisfying that $\|\tilde{f}\|_{H^\frac{1}{2}(S_1)} \leq C \|f\|_{H^\frac{1}{2}(S_1,D)}$ with $C$ independent of $f$ and let $u_0 \in H^1(BR \setminus \Omega_2)$ be such that $u_0 = f$ on $S_1$, $u_0 = 0$ on $\partial BR$ and $\|u_0\|_{H^1(BR \setminus \Omega_2)} \leq C \|\tilde{f}\|_{H^\frac{1}{2}(S_1)}$ (this is possible in view of theorem 3.37 in [19]). Then for every solution $u$ to the problem (2.1)–(2.5) and (2.8), $w = u - u_0$ is in the Sobolev space $H^1_0(BR \setminus \Omega_2, S_{1,1})$ defined by 
\[H^1_0(BR \setminus \Omega_2, S_{1,1}) := \{w \in H^1(BR \setminus \Omega_2) : w = 0 \text{ on } S_{1,1}\}.
\]

Multiplying equations (2.1) and (2.2) by a test function $\varphi \in H^1_0(BR \setminus \Omega_2, S_{1,1})$, integrating by parts and using the boundary conditions on $S_1$, $S_0$, $\partial BR$, we obtain the following variational formulation for the problem (2.1)–(2.5) and (2.8): find $w \in H^1_0(BR \setminus \Omega_2, S_{1,1})$ such that 
\[a(w, \varphi) = -a(u_0, \varphi) + b(\varphi) \quad \forall \varphi \in H^1_0(BR \setminus \Omega_2, S_{1,1})
\]
where, for $v, \varphi \in H^1_0(BR \setminus \Omega_2, S_{1,1})$,
\[a(v, \varphi) = \lambda_0 \int_{\Omega_1} (\nabla v \cdot \nabla \varphi - k^2 v \varphi) \, dx + \int_{B_{\lambda} \setminus \Omega} (\nabla v \cdot \nabla \varphi - k^2_0 v \varphi) \, dx
\]
\[\quad - \int_{\partial B_{\lambda}} T v \varphi \, ds - i\lambda_0 \int_{S_{1,1}} \lambda v \varphi \, ds,
\]
and 
\[b(\varphi) = -\lambda_0 \int_{\Omega_1} h \varphi \, dx - \int_{S_0} g \varphi \, ds - \lambda_0 \int_{S_{1,1}} p \varphi \, ds.
\]
We write $a = a_1 + a_2$ with
\[a_1(v, \varphi) = \lambda_0 \int_{\Omega_1} (\nabla v \cdot \nabla \varphi + v \varphi) \, dx + \int_{B_{\lambda} \setminus \Omega} (\nabla v \cdot \nabla \varphi + v \varphi) \, dx
\]
\[\quad - \int_{\partial B_{\lambda}} T_0 v \varphi \, ds - i\lambda_0 \int_{S_{1,1}} \lambda v \varphi \, ds
\]
and 
\[a_2(v, \varphi) = -\lambda_0 (1 + k^2_0) \int_{\Omega_1} v \varphi \, dx - (1 + k^2_0) \int_{B_{\lambda} \setminus \Omega} v \varphi \, dx - \int_{\partial B_{\lambda}} (T - T_0) v \varphi \, ds,
\]
where $T_0$ is the operator defined in lemma 2.4. By the boundedness of $T_0$ and the trace theorem, $a_1$ is bounded. On the other hand, for all $w \in H^1_0(BR \setminus \Omega_2, S_{1,1})$,
\[\Re a_1(w, w) \geq \min(\lambda_0, 1) \|w\|_{H^1_0(BR \setminus \Omega_2, S_{1,1})}^2 - \int_{\partial B_{\lambda}} T_0 w \varphi \, ds
\]
\[\geq \min(\lambda_0, 1) \|w\|_{H^1_0(BR \setminus \Omega_2, S_{1,1})}^2 + C \|w\|_{H^\frac{1}{2}(\partial B_{\lambda})}^2
\]
\[\geq \min(\lambda_0, 1) \|w\|_{H^1_0(BR \setminus \Omega_2, S_{1,1})}^2,
\]
that, is, $a_1$ is strictly coercive. By the compactness of $T-T_0$ and Rellich’s selection theorem
(that is, the compact imbedding of $H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D)$ into $L^2(B_R \setminus \overline{\Omega}_2)$), it follows that $a_2$
is compact. The Lax–Milgram theorem implies that there exists a bijective bounded linear
operator $A : H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D) \to H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D)$ satisfying

$$a_1(w, \varphi) = (Aw, \varphi) \quad \text{for all} \quad \varphi \in H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D).$$

By the Riesz representation theorem, there exists a bounded linear operator
$B : H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D) \to H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D)$ such that

$$a_2(w, \varphi) = (Bw, \varphi) \quad \text{for all} \quad \varphi \in H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D).$$

By the Riesz representation theorem again, one can find $v \in H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D)$ such that

$$F(\varphi) := -a(u_0, \varphi) + b(\varphi) = (v, \varphi) \quad \text{for all} \quad \varphi \in H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D).$$

Thus, the variational formulation (2.10) is equivalent to the problem:

Find $w \in H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D)$ such that $Aw + Bw = v,$ \hspace{1em} (2.11)

where $A$ is bounded and strictly coercive and $B$ is compact. The Riesz–Fredholm theory
and the uniqueness result (theorem 2.3) imply that the problem (2.11) or equivalently the problem (2.10) has a unique solution. The estimate (2.9) follows from the fact that, by a duality
argument, $\|v\|_{H_0^1(B_R \setminus \overline{\Omega}_2, S_1, D)} = \|F\|$ is bounded by $\|p\|_{H^{-1}(S_0)}$, $\|h\|_{L^2(\Omega_1)}$, $\|g\|_{H^{-1}(S_0)}$ and $\|u_0\|_{H^1(B_R \setminus \overline{\Omega}_2)}$ which in turn is bounded by $\|f\|_{H^{1/2}(S_0)}$. \hspace{1em} □

3. The inverse scattering problem

To establish the uniqueness result for the inverse scattering problem as mentioned in the
introduction, we need a generalization of the mixed reciprocity relation.

The scattering of incident plane waves in a two-layered background medium can be formulated as follows:

$$\Delta u + k_j^2 u = 0 \quad \text{in} \quad \Omega_0, \hspace{1em} \text{(3.1)}$$

$$\Delta u + k_l^2 u = 0 \quad \text{in} \quad \Omega_1, \hspace{1em} \text{(3.2)}$$

$u_+ = u_-, \quad \frac{\partial u_+}{\partial v} = \lambda_0 \frac{\partial u_-}{\partial v} \quad \text{on} \quad S_0, \hspace{1em} \text{(3.3)}$

$$B(u) = 0 \quad \text{on} \quad S_1, \hspace{1em} \text{(3.4)}$$

$$u = u^i + u^r \quad \text{in} \quad \Omega_0 \cup \Omega_1, \hspace{1em} \text{(3.5)}$$

$$\lim_{r \to \infty} r^{1/2} \left( \frac{\partial u^r}{\partial r} - ik_0 u^r \right) = 0, \quad r = |x|, \hspace{1em} \text{(3.6)}$$

with positive wave numbers $k_j \quad (j = 0, 1)$ and the nonnegative constant $\lambda_0$.

Recall that the fundamental solution of the Helmholtz equation is given by

$$\Phi(x, y) = \begin{cases} \frac{e^{ik_0|x-y|}}{4\pi|x-y|} & \text{for} \quad x, y \in \mathbb{R}^3, \quad x \neq y, \\ \frac{i}{4} \cdot H_0^1(k_0|x-y|) & \text{for} \quad x, y \in \mathbb{R}^2, \quad x \neq y, \end{cases}$$

where $H_0^1$ is the Hanker function of the first kind of order zero.

As incident fields $u^i$, plane waves and point sources are of special interest. Denote by $u^r(\cdot, d)$ the scattered field for an incident plane wave $u^i(\cdot, d)$ with the incident direction $d \in S$. 

and by $u^\infty(\cdot, d)$ the corresponding far-field pattern. The scattered field for an incident point source $\Phi(\cdot, z)$ with the source point $z \in \mathbb{R}^n$ is denoted by $u'(\cdot; z)$ and the corresponding far-field pattern by $\Phi^\infty(\cdot, z)$.

**Remark 3.1.** The trivial field $u = 0$ is not a solution of the scattering problem in the case when the incident field is a point source $\Phi(\cdot, z)$ with the source point $z \in \mathbb{R}^n$—although it satisfies the radiation condition (since the incident field does). This is because in this case the scattered field $u'(\cdot; z) = -\Phi(\cdot, z)$ has a singularity at the source point $z \in \mathbb{R}^n$, contradicting to the fact that the scattered field has no singularities. In fact, in this case, equation (3.1) or (3.2) above should hold in $\Omega_0 \setminus \{z\}$ or $\Omega_1 \setminus \{z\}$ depending on where the source point $z$ is (cf (3.17) in lemma 3.5 below).

For the mixed reciprocity relation we need the constant

$$
\gamma_n = \begin{cases} 
\frac{1}{4\pi}, & n = 3, \\
\frac{\sqrt{8\pi n}}{\pi}, & n = 2 
\end{cases}
$$

depending on the dimension $n$.

**Lemma 3.2.** (*Mixed reciprocity relation.*) For the scattering of plane waves $u'(\cdot, d)$ with $d \in S$ and point sources $\Phi(\cdot, z)$ from an obstacle $\Omega_2$ we have

$$
\Phi^\infty(\hat{x}, z) = \begin{cases} 
\gamma_n u^s(z, -\hat{x}), & z \in \Omega_0, \hat{x} \in S, \\
\lambda_0 \gamma_n u^s(z, -\hat{x}) + (\lambda_0 - 1) \gamma_n u^s(z, -\hat{x}), & z \in \Omega_1, \hat{x} \in S. 
\end{cases}
$$

**Remark 3.3.** The proof for the case of an obstacle in a homogeneous medium (i.e. the case $\Omega = \Omega_2$) can be found in [17] or [21].

**Proof.** By Green’s second theorem and the Sommerfeld radiation condition we have that

$$
\int_{S_0} \left( u^s(y; z) \frac{\partial u^s(y, d)}{\partial \nu(y)} - u^s(y, d) \frac{\partial u^s(y; z)}{\partial \nu(y)} \right) ds(y) = 0 \quad (3.7)
$$

for $z \in \Omega_0 \cup \Omega_1$ and $d \in S$. Using theorem 2.5 in [8], we obtain the representation

$$
\Phi^\infty(\hat{x}, z) = \gamma_n \int_{S_0} \left( u^s(y; z) \frac{\partial e^{-ik_0 y \cdot \hat{x}}}{\partial v(y)} - e^{-ik_0 y \cdot \hat{x}} \frac{\partial u^s(y; z)}{\partial v(y)} \right) ds(y) \quad (3.8)
$$

for $z \in \Omega_0 \cup \Omega_1$ and $\hat{x} \in S$.

We first consider the case $z \in \Omega_0$. Adding equation (3.8) to equation (3.7) multiplied by $\lambda_0$ and with $d$ replaced by $-\hat{x}$ to equation (3.8) we obtain, with the help of the boundary condition on $S_0$ and Green’s second theorem, that for $z \in \Omega_0$, $\hat{x} \in S$,

$$
\Phi^\infty(\hat{x}, z) = \gamma_n \int_{S_0} \left( u^s(y; z) \frac{\partial u^s(y, -\hat{x})}{\partial v(y)} - u^s(y, -\hat{x}) \frac{\partial u^s(y; z)}{\partial v(y)} \right) ds(y) \\
= \lambda_0 \gamma_n \int_{S_1} \left( u^s(y; z) \frac{\partial u^s(y, -\hat{x})}{\partial v(y)} - u^s(y, -\hat{x}) \frac{\partial u^s(y; z)}{\partial v(y)} \right) ds(y) \\
+ \lambda_0 \gamma_n \int_{\Omega_1} \left( (k_1^2 - k_0^2) \Phi(y, z) u(y, -\hat{x}) \right) dy \\
+ (1 - \lambda_0) \gamma_n \int_{S_0} \left( u^s(y, -\hat{x}) \frac{\partial \Phi(y, z)}{\partial v(y)} \right) ds(y). \quad (3.9)
$$
Green’s second theorem gives that for \( z \in \Omega_0 \) and \( \tilde{x} \in S 
abla \gamma_n \int_{S} u^l(y, -\tilde{x}) \frac{\partial \Phi(z, y)}{\partial v(y)} - \Phi(z, y) \frac{\partial u^l(y, -\tilde{x})}{\partial v(y)} \, ds(y) = 0. \) (3.10)

Using Green’s formula (see theorem 2.4 in [8]) we have
\[
\gamma_n u^l(z, -\tilde{x}) = \gamma_n \int_{S} \left( u^l_s(y, -\tilde{x}) \frac{\partial \Phi(z, y)}{\partial v(y)} - \Phi(z, y) \frac{\partial u^l_s(y, -\tilde{x})}{\partial v(y)} \right) \, ds(y)
\]
(3.11) for \( z \in \Omega_0 \) and \( \tilde{x} \in S \). Adding (3.10) to equation (3.11) we deduce, with the help of the boundary condition on \( S_0 \) and Green’s second theorem, that
\[
\gamma_n u^l(z, -\tilde{x}) = \gamma_n \int_{S} \left( u^l_s(y, -\tilde{x}) \frac{\partial \Phi(z, y)}{\partial v(y)} - \Phi(z, y) \frac{\partial u^l_s(y, -\tilde{x})}{\partial v(y)} \right) \, ds(y) \\
= \lambda_0 \gamma_n \int_{S_1} \left( u^l_s(y, -\tilde{x}) \frac{\partial \Phi(z, y)}{\partial v(y)} - \Phi(z, y) \frac{\partial u^l_s(y, -\tilde{x})}{\partial v(y)} \right) \, ds(y) \\
+ \lambda_0 \gamma_n \int_{\Omega_1} \left( (k^2_l - k^2_0) \Phi(y, z) u(y, -\tilde{x}) \right) \, dy \\
+ (1 - \lambda_0) \gamma_n \int_{S_0} \left( u^l_s(y, -\tilde{x}) \frac{\partial \Phi(z, y)}{\partial v(y)} \right) \, ds(y)
\] (3.12)

for \( z \in \Omega_0 \) and \( \tilde{x} \in S \). It is easy to see that the last two terms on the right-hand side (rhs) of (3.9) and (3.12) coincide. By the boundary condition on \( S_1 \), the first terms on the rhs of (3.9) and (3.12) coincide, so \( \Phi^\infty(\tilde{x}, z) = \gamma_n u^l(z, -\tilde{x}) \) for all \( z \in \Omega_0 \), \( \tilde{x} \in S \).

We now consider the case \( z \in \Omega_1 \). Adding equation (3.8) to equation (3.7) multiplied by \( \gamma_n \) and with \( d \) replaced by \(-\tilde{x}\) to equation (3.8) and using the boundary condition on \( S_0 \) and Green’s second theorem yield that
\[
\Phi^\infty(\tilde{x}, z) = \gamma_n \int_{S_1} \left( u^l_s(y; z) \frac{\partial u^l_s(y, -\tilde{x})}{\partial v(y)} - u^s(y, -\tilde{x}) \frac{\partial u^l_s(y; z)}{\partial v(y)} \right) \, ds(y).
\]
We circumscribe the point \( z \in \Omega_2 \) with a sphere \( \Omega(z; \epsilon) := \{ y \in \mathbb{R}^d : |y - z| = \epsilon \} \) contained in \( \Omega_1 \). Applying Green’s second theorem in the domain \( \Omega_1, \epsilon := \{ y \in \Omega_1 : |y - z| > \epsilon \} \) and taking into account the boundary condition on \( S_0 \) we obtain that
\[
\Phi^\infty(\tilde{x}, z) = \lambda_0 \gamma_n \int_{S_1} \left( u^l_s(y; z) \frac{\partial u^l_s(y, -\tilde{x})}{\partial v(y)} - u^s(y, -\tilde{x}) \frac{\partial u^l_s(y; z)}{\partial v(y)} \right) \, ds(y) \\
+ \lambda_0 \gamma_n \int_{\Omega(z; \epsilon)} \left( u^l_s(y; z) \frac{\partial u^l_s(y, -\tilde{x})}{\partial v(y)} - u^s(y, -\tilde{x}) \frac{\partial u^l_s(y; z)}{\partial v(y)} \right) \, ds(y) \\
+ \lambda_0 \gamma_n \int_{\Omega_1} \left( (k^2_l - k^2_0) \Phi(y, z) u(y, -\tilde{x}) \right) \, dy \\
+ (1 - \lambda_0) \gamma_n \int_{S_0} \left( u^l_s(y, -\tilde{x}) \frac{\partial \Phi(z, y)}{\partial v(y)} \right) \, ds(y)
\] (3.13)

for \( z \in \Omega_1 \) and \( \tilde{x} \in S \). By the well posedness of the direct problem and the interior elliptic regularity (see section 6.3.1 in [12]), \( u^l(\cdot, -\tilde{x}) \in C^\infty(\Omega_1) \) and \( u^l(\cdot, z) \in H^2(V) \) for any compact subset \( V \) of \( \Omega_1 \). Thus, there is a sequence \( \epsilon_j \) such that \( \epsilon_j \to 0 \) and
\[
\int_{\Omega(z; \epsilon_j)} \left( |u^l(\cdot, z)^2| + |\nabla u^l(\cdot, z)|^2 \right) \, ds(y) \to 0
\]
as \( j \to \infty \). This together with the Cauchy–Schwarz inequality implies that the integral on \( \Omega(z; \epsilon) \) with \( \epsilon = \epsilon_j \) tends to 0 as \( j \to \infty \). By passing to the limit \( j \to \infty \) in (3.13) with
\( \epsilon = \epsilon_j \) we have

\[
\Phi^\infty(\hat{x}, z) = \lambda_0 \gamma_n \int_{S_i} \left( u_+(y; \hat{x}, z) \frac{\partial u_+(y; \hat{x}, z)}{\partial v(y)} - u_+(y; \hat{x}, z) \frac{\partial \Phi(y, z; \hat{x})}{\partial v(y)} \right) \, ds(y)
\]

\[
+ \lambda_0 \gamma_n \int_{\Omega_i} \left( (k_1^2 - k_0^2) \Phi(y, z) u(y, \hat{x}) \right) \, dy
\]

\[
+ (1 - \lambda_0) \gamma_n \int_{S_0} \left( u_-(y, \hat{x}) \frac{\partial \Phi(y, z; \hat{x})}{\partial v(y)} \right) \, ds(y).
\]

The volume integral exists as an improper integral since its integrand is weakly singular.

On the other hand, with the help of Green’s formula (see theorem 2.1 in [8]), we have

\[
u(z, \hat{x}) = \int_{S_0} \left( \Phi(z, y) \frac{\partial u_-(y, \hat{x})}{\partial v(y)} - u_-(y, \hat{x}) \frac{\partial \Phi(z, y)}{\partial v(y)} \right) \, ds(y)
\]

\[
- \int_{S_i} \left( \Phi(z, y) \frac{\partial u_+(y, \hat{x})}{\partial v(y)} - u_+(y, \hat{x}) \frac{\partial \Phi(z, y)}{\partial v(y)} \right) \, ds(y)
\]

\[
+ \int_{\Omega_i} \left( (k_1^2 - k_0^2) \Phi(z, y) u(y, \hat{x}) \right) \, dy.
\]

It follows from (3.14) and (3.15) together with the boundary conditions on \( S_0 \) and \( S_1 \) and Green’s second theorem that

\[
\Phi^\infty(\hat{x}, z) = \lambda_0 \gamma_n u^\prime(z, \hat{x}) + (\lambda_0 - 1) \gamma_n u^\prime(z, \hat{x}) \text{ for all } z \in \Omega_1, \hat{x} \in S.
\]

Remark 3.4. For the case of an \((N + 1)\)-layered background medium, it is easy to deduce that

\[
\Phi^\infty(\hat{x}, z) = \begin{cases} 
\gamma_n u^\prime(z, \hat{x}), & z \in \Omega_0, \hat{x} \in S, \\
\lambda_0 \lambda_1 \cdots \lambda_{N-1} \gamma_n u^\prime(z, \hat{x}) + (\lambda_0 \lambda_1 \cdots \lambda_{N-1} - 1) \gamma_n u^\prime(z, \hat{x}), & z \in \Omega_N, \hat{x} \in S.
\end{cases}
\]

Lemma 3.5. For \( n = 2 \) or 3 and for \( \Omega_2 \subset \Omega \) let \( G \) be the unbounded component of \( \mathbb{R}^n \setminus (\Omega_2 \cup \overline{\Omega_2}) \) and let \( u^\infty(\hat{x}, d) = \tilde{u}^\infty(\hat{x}, d) \) for all \( \hat{x}, d \in S \) with \( \tilde{u}^\infty(\hat{x}, d) \) being the far-field pattern of the scattered field \( \tilde{u}^i(x, d) \) corresponding to the obstacle \( \Omega_2 \) and the same incident plane wave \( u^i(x, d) \). For \( z_1 \in \Omega \cap \Omega_2 \) let \( u^\prime = u^\prime(x; z_1) \) be the unique solution of the problem

\[
\Delta u^\prime + k_0^2 u^\prime = 0 \quad \text{in} \quad \Omega_0,
\]

\[
\Delta u^\prime + k_0^2 u^\prime = (k_0^2 - k_1^2) \Phi(x, z_1) \quad \text{in} \quad \Omega_1 \setminus \{z_1\},
\]

\[
u^\prime = u^\prime, \quad \frac{\partial u^\prime}{\partial v} - \lambda_0 \frac{\partial u^\prime}{\partial v} = (\lambda_0 - 1) \frac{\partial \Phi(x, z_1)}{\partial v} \quad \text{on} \quad S_0,
\]

\[
B(u^\prime) = -B(\Phi(x, z_1)) \quad \text{on} \quad S_1,
\]

\[
\Delta u^\prime + k_0^2 u^\prime = 0 \quad \text{in} \quad \Omega_0,
\]

\[
\Delta u^\prime + k_0^2 u^\prime = (k_0^2 - k_1^2) \Phi(x, z_1) \quad \text{in} \quad \Omega_1 \setminus \{z_1\},
\]

\[
u^\prime = u^\prime, \quad \frac{\partial u^\prime}{\partial v} - \lambda_0 \frac{\partial u^\prime}{\partial v} = (\lambda_0 - 1) \frac{\partial \Phi(x, z_1)}{\partial v} \quad \text{on} \quad S_0,
\]

\[
B(u^\prime) = -B(\Phi(x, z_1)) \quad \text{on} \quad S_1,
\]
Assume that \( \tilde{u} = \tilde{u}(x; z_1) \) is the unique solution of the problem (3.16)–(3.20) with \( \Omega_2 \) replaced by \( \tilde{\Omega}_2 \) and \( \Omega_1 \) replaced by \( \tilde{\Omega}_1 := \Omega \setminus \tilde{\Omega}_2 \). Then we have

\[
\tilde{u}'(x; z_1) = \tilde{u}'(x; z_1), \quad x \in \tilde{G}.
\]

(3.21)

**Remark 3.6.** By Theorem 2.5, the problem (3.16)–(3.20) has a unique solution.

**Proof.** By Rellich’s lemma \([8]\), the assumption \( u^\infty(\hat{x}, d) = \tilde{u}^\infty(\hat{x}, d) \) for all \( \hat{x}, \ d \in S \) implies that

\[
u(x, \hat{v}, d) \Rightarrow \nu(x, \hat{v}, d), \quad \frac{\partial u^\infty(x, d)}{\partial v} = \frac{\partial \tilde{u}^\infty(x, d)}{\partial v}, \quad x \in S_0, \quad d \in S.
\]

Using Holmgren’s uniqueness theorem \([16]\), we obtain that \( u^\prime(z_1, d) = \tilde{u}^\prime(z_1, d) \) for \( z_1 \in \Omega \cap G, \ d \in S \). For the far-field pattern of incident point sources we have by lemma 3.2 that

\[
\Phi^\infty(d, z_1) = \tilde{\Phi}^\infty(d, z_1), \quad z_1 \in \Omega \cap G, \quad d \in S.
\]

Thus, Rellich’s lemma \([8]\) gives

\[
u(x, \hat{v}, d) \Rightarrow \nu(x, \hat{v}, d), \quad \frac{\partial u^\prime(x, z_1)}{\partial v} = \frac{\partial \tilde{u}^\prime(x, z_1)}{\partial v}, \quad x \in S_0, \quad z_1 \in \Omega \cap G.
\]

From Holmgren’s uniqueness theorem \([16]\) it is derived that

\[
u(x, \hat{v}, d) \Rightarrow \nu(x, \hat{v}, d), \quad x \in \Omega \cap G, \quad z_1 \in \Omega \cap G,
\]

which yields the desired result (3.21).

Using the generalized mixed reciprocity relation (lemma 3.2) and lemma 3.5, the following uniqueness result can be proved following Colton and Kress \([9]\) and Kress \([18]\).

**Theorem 3.7.** Assume that \( \Omega_2 \) and \( \tilde{\Omega}_2 \) are two obstacles with boundary conditions \( B \) and \( \tilde{B} \), respectively, for the same piecewise homogeneous background medium. If the far-field patterns of the scattered fields for the same incident plane wave \( u^\prime(x) = e^{ik_0 x \cdot d} \) coincide at a fixed frequency for all incident direction \( d \in S \) and observation direction \( \hat{x} \in S \), then \( \Omega_2 = \tilde{\Omega}_2, \ B = \tilde{B} \).

**Proof.** Let \( G \) be the unbounded component of \( \mathbb{R}^n \setminus (\tilde{\Omega}_2 \cup \Omega_2) \). Assume that \( \Omega_2 \neq \tilde{\Omega}_2 \). Then, without loss of generality, we may assume that there exists \( z_0 \in \partial \tilde{\Omega}_2 \cap (\mathbb{R}^n \setminus \Omega_2) \). Choose \( h > 0 \) such that the sequence

\[
z_j := z_0 + \frac{h}{j} v(z_0), \quad j = 1, 2, \ldots,
\]

is contained in \( \Omega \cap G \), where \( v(z_0) \) is the outward normal to \( \partial \Omega_2 \) at \( z_0 \). Consider the solution to the problem (3.16)–(3.20) with \( z_1 \) replaced by \( z_j \). By lemma 3.5, we have that

\[
u(\hat{x}, \hat{v}, d) \Rightarrow \nu(\hat{x}, \hat{v}, d), \quad x \in G \setminus \{z_j\}.
\]

Since \( z_0 \) has a positive distance from \( \tilde{\Omega}_2 \), we conclude from the well posedness of the direct scattering problem that there exists \( C > 0 \) such that

\[
|B(\tilde{u}^\prime(z_0; z_j))| \leq C
\]

uniformly for \( j \geq 1 \). On the other hand, by the boundary condition on \( \partial \Omega_2 \),

\[
|B(\tilde{u}^\prime(z_0; z_j))| = |B(u^\prime(z_0; z_j))| = |B(\Phi(z_0, z_j))| \to \infty
\]
as \( j \to \infty \). This is a contradiction, which implies that \( \Omega_2 = \tilde{\Omega}_2 \).
We now assume that the boundary conditions are different, that is $B \neq \tilde{B}$. First, for the case of impedance boundary conditions we assume that we have two different continuous impedance functions $\lambda \neq \tilde{\lambda}$. Then from the conditions
\[
\frac{\partial u}{\partial \nu} + i\lambda u = 0, \quad \frac{\partial u}{\partial \nu} + i\tilde{\lambda} u = 0 \quad \text{on} \quad S_1,
\]
we obtain that
\[
(\lambda - \tilde{\lambda}) u = 0 \quad \text{on} \quad S_1.
\]
Therefore, on the open set $\Gamma := \{ x \in S_1 : \lambda \neq \tilde{\lambda} \}$ we have that $\frac{\partial u}{\partial \nu} = u = 0$. Then Holmgren’s uniqueness theorem [16] implies that the total field $u = u^i + u^s = 0$. The scattered field $u^s$ tends to zero uniformly at infinity while the incident plane wave has modulus 1 everywhere. Thus, the modulus of the total field tends to 1. This leads to a contradiction, giving that $\lambda = \tilde{\lambda}$. The case for other boundary conditions can be dealt with similarly. □

Acknowledgments

This work was partly supported by the Chinese Academy of Sciences through the Hundred Talents Program and by the NNSF of China under grant No 10671201. The authors would like to thank the referees for their invaluable comments which helped improve the presentation of the paper.

References


[24] Yaman F 2009 Location and shape reconstruction of sound soft obstacle buried in penetrable cylinders *Inverse Problems* 25 065005
