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Inverse wave scattering by unbounded obstacles: uniqueness for the two-dimensional Helmholtz equation

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In this article we present some uniqueness results on inverse wave scattering by unbounded obstacles for the two-dimensional Helmholtz equation. We prove that an impenetrable one-dimensional rough surface can be uniquely determined by the values of the scattered field taken on a line segment above the surface that correspond to the incident waves generated by a countable number of point sources. For penetrable rough layers in a piecewise constant medium, the refractive indices together with the rough interfaces (on which the TM transmission conditions are imposed) can be uniquely identified using the same measurements and the same incident point source waves. Moreover, a Dirichlet polygonal rough surface can be uniquely determined by a single incident point source wave provided a certain condition is imposed on it.

Keywords: inverse scattering; uniqueness; rough surface; Helmholtz equation; point sources

AMS Subject Classifications: 35R30; 78A46

1. Introduction

Inverse rough surface scattering problems have many applications in micro-optics, radar imaging and non-destruction testing. For instance, the determination of the elevation of the ground, sea surface or sea bed are basic problems in remote sensing by sonar or radar. This article is concerned with the uniqueness in inverse wave scattering problems for penetrable or impenetrable unbounded obstacles which can be modelled by the two-dimensional Helmholtz equation.

There have been several uniqueness results on inverse diffraction problems for both penetrable and impenetrable periodic structures, which can be viewed as a special case of unbounded rough surfaces. For the inverse Dirichlet problem with a C^2-smooth periodic boundary, we refer to Bao [1] in the case of a lossy medium (i.e. $\operatorname{Im} k > 0$), Kirsch [2] for using all quasi-periodic incident waves, and Hettlich and

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Kirsch [3] for sufficiently small wave number or grating height and one incident plane wave. In the case of electromagnetic scattering in the TE mode by one periodic interface, Elschner and Yamamoto [4] proved that measurements corresponding to a finite number of refractive indices above or below the grating profile uniquely determine the periodic interface. This extends the uniqueness result by Hettlich and Kirsch based on Schiffer's theorem [3] to the inverse transmission problem. For two periodic interfaces with an inhomogeneity between them, it was proved in [5] that the interfaces and transmission coefficients can be uniquely identified from the scattered fields for all quasi-periodic incident waves, and so can the refractive index of the inhomogeneity if it only depends on x_1 and the interfaces are parallel to the x_1-axis. Note that the measurements in [4,5] must be taken both above and below the structure.

The mathematical theory of forward scattering by an unbounded rough surface was mainly established by Chandler-Wilde and his collaborators over the last 15 years, using integral equation methods (see, e.g. [6–8]) or variational methods [9,10]. We also refer to [11] for the uniqueness issue that occured in the direct problem for exterior acoustics. Concerning uniqueness in inverse rough surface scattering problem, as far as we know, the only reference is due to Chandler-Wilde and Ross [12] who proved that a Dirichlet rough surface in a lossy medium can be uniquely determined by the near-field data above the surface corresponding to only one incident plane wave, which generalizes Bao's result [1] on periodic structures to rough surface scattering.

If the wave number k is a real number, it is well known that global uniqueness in determining a Dirichlet surface is impossible in general with only one incident plane wave [1,3]. Moreover, it is shown in [13] that, for each incident plane wave, there exist two classes of polygonal periodic structures which cannot be uniquely determined, one of which is the set of straight lines parallel to the x_1-axis. Non-uniqueness examples can be readily constructed from these two unidentifiable classes provided the incident angle and the wave number satisfy certain relations.

In this article we present new uniqueness results using the incident waves generated by point sources. In Section 2.1, we prove that a Dirichlet rough surface can be uniquely determined by near-field data on a line segment above the surface corresponding to a countable number of incident point source waves, following the approach of Kirsch and Kress [14] for bounded obstacles and that of Kirsch [2] for periodic structures. However, when rough surfaces are confined to polygonal periodic structures, the measurements for one incident point source wave are sufficient to ensure uniqueness. The proof in the latter case is based on the reflection principle for the Helmholtz equation and the reduction argument from [15]. Such a uniqueness result also applies to non-periodic rough polygonal surfaces satisfying certain conditions; see Section 2.3. Finally, in Section 3 we extend the argument of Section 2.1 to the TM transmission problem for penetrable rough layers in a piecewise constant medium. This is motivated by our recent work [16] on inverse scattering by multilayered bounded obstacles and periodic structures.

In Sections 2.1 and 3 we always assume that the non-periodic rough surface or interface is given by the graph of a $C^{1,1}$ function. This regularity assumption can be relaxed in Section 2.1 (see, e.g. [9,10] for the direct problem), while it is very necessary in Section 3 for the inverse transmission problem in order to tackle the singularity of the fundamental solution in a half-space.

2. Uniqueness for the Dirichlet problem

In this section, we consider uniqueness in inverse wave scattering by an impenetrable rough surface on which the Dirichlet boundary condition is satisfied. Such a uniqueness issue arises in acoustic wave scattering by sound-soft unbounded obstacles and electromagnetic scattering in the TE mode by an unbounded perfect conductor.

2.1. *Uniqueness for general rough surfaces*

We begin with some mathematical formulations and solvability results on the forward scattering problem, and then precisely formulate the inverse Dirichlet problem. For $H\in\mathbb{R}$, let $U_H=\{x_2>H\}$ and $\Gamma_H=\{x_2=H\}$. Let $C^{1,1}(\mathbb{R})$ denote the set of functions f: $\mathbb{R}\to\mathbb{R}$ which are bounded and continuously differentiable, with Lipschitz continuous derivative. Given a function $f\in C^{1,1}(\mathbb{R})$, which satisfies, for some constants $f_+>f_->0$,

$$f_- < f(x_1) < f_+, \quad x_1\in\mathbb{R},$$

we define the two-dimensional region D by

$$D := \{x = (x_1, x_2)\in\mathbb{R}^2 : x_2 > f(x_1)\}. \tag{1}$$

Assume that the one-dimensional scattering rough surface Λ is given by

$$\Lambda := \partial D = \{(x_1, f(x_1)) : x_1\in\mathbb{R}\}, \tag{2}$$

and that an incident wave $u^{in}(x;z)$ generated by point source $z\in D$,

$$u^{in}(x;z) := (i/4)H_0^{(1)}(k|x-z|), \tag{3}$$

is incident on Λ from the top region D, with $H_0^{(1)}(t)$ being the Hankel function of the first kind of order zero. The above defined incident wave $u^{in}(x;z)$ is nothing else but the fundamental solution to the Helmholtz equation $(\Delta+k^2)u=0$ in the whole two-dimensional space, and is also referred to as the incident point source wave throughout this article. It is supposed that the wave number k is a positive constant, and that the total field $u(x;z)$, which is the sum of the incident field $u^{in}(x;z)$ and the corresponding scattered field $u^{sc}(x;z)$, vanishes on the boundary Λ of D.

Since the region D is unbounded in x_2, a radiation condition as $x_2\to+\infty$ has to be imposed on the scattered field. We adopt the upward propagating radiation condition (UPRC), first proposed by Chandler-Wilde and Zhang [8], to represent u^{sc} explicitly in the upper half-space U_H for some $H>f_+$ via its Dirichlet value $u^{sc}|_{\Gamma_H}$, i.e.

$$u^{sc}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(i[(x_2-H)\sqrt{k^2-\xi^2}+x_1\xi]\right)\hat{F}_H(\xi)\mathrm{d}\xi, \quad x\in U_H, \tag{4}$$

where $\sqrt{k^2-\xi^2}=i\sqrt{\xi^2-k^2}$ when $|\xi|>k$, and $\hat{F}_H$ denotes the Fourier transformation of $u^{sc}(x_1,H)$ with respect to x_1 defined by

$$\hat{F}_H(\xi) := \mathcal{F}(u^{sc}(x_1,H))(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(-ix_1\xi)u^{sc}(x_1,H)\mathrm{d}x_1, \quad \xi\in\mathbb{R}.$$

The integral in (4) exists in the Lebesgue sense provided $u^{sc}(x_1, H)$ belongs to $L^2(\mathbb{R})$ so that $\hat{F}_H$ belongs to $L^2(\mathbb{R})$. The above UPRC is also referred to as the angular spectrum representation in the literature, and is equivalent to the pole condition based on the Laplace transform of the solution in the radial direction. We refer to Arens and Hohage [17] for the details about this equivalence, and the interpretation of the integral (4) if $u^{sc}(x_1, H)$ is a bounded continuous function. In addition, we remark that the UPRC does not depend on the choice of $H > f_+$ [10, Remark 2.1], and generalizes the standard Rayleigh expansion condition for periodic structures [6]. From (4), we observe that u^{sc} is the linear superposition of the upward propagating plane waves $\exp(i(x_2 - H)\sqrt{k^2 - \xi^2} + ix_1\xi)$ for $|\xi| \le k$, and the evanescent surface waves $\exp(-(x_2 - H)\sqrt{\xi^2 - k^2} + ix_1\xi])$ for $|\xi| > k$.

Now, we can formulate the direct and inverse scattering problems as follows.

(DP) Given Λ and the incident wave $u^{in}(x; z)$ for some $z \in D$, determine the total field $u = u^{in} + u^{sc}$ such that

$$(\Delta + k^2)u = -\delta(x - z) \quad \text{in } D, \quad u = 0 \quad \text{on } \Lambda,$$

and such that $u^{sc}(x; z) \in C^2(D) \cap C(\overline{D})$ satisfies the UPRC and $\sup_{x \in D} x_2^{\beta} |u^{sc}(x)| < +\infty$ for some $\beta \in \mathbb{R}$.

Using the integral equation method, it is shown in Chandler-Wilde et al. [7, Theorem 5.3] that the problem (DP) is uniquely solvable provided $f \in C^{1,1}$, with the estimate

$$|u^{sc}(x)| \le C x_2^{1/2} \|u^{in}\|_{L^\infty(\Lambda)}$$

for some constant $C > 0$ independent of the incident field. Recently, Elschner and Chandler-Wilde [9] were able to prove the well-posedness of (DP) using the variational method in weighted Sobolev spaces for much more general boundaries. Since surface waves of the scattered field can be hardly detected far away from the rough surface, the inverse problem always involves in near-field measurements. Given $b > f_+$ and $b^* > 0$, define the line segment Γ_b^* by

$$\Gamma_b^* := \{(x_1, b) : |x_1| < b^*\}.$$

We proceed with the inverse problem (IP):

(IP) Given one incident point source wave $u^{in}(x; y)$ for some $y \in U_{f_+}$, determine the rough surface Λ from the knowledge of the near-field data $\{u(x; y) : x \in \Gamma_b^*\}$.

Remark 1 If the incident point source wave is replaced with a plane wave, then the uniqueness to (IP) does not hold if $k > 0$. It is proved in [13] that two incident plane waves are always sufficient to uniquely determine a non-flat polygonal periodic structure under the Dirichlet boundary condition, while a straight line parallel to the x_1-axis cannot be uniquely determined by a finite number of incident plane waves in general. Nevertheless, if the medium in D is lossy, i.e. Im $k > 0$, Chandler-Wilde and Ross [6] proved that uniqueness to (IP) holds true with one incident plane wave.

As far as we know, the uniqueness to (IP) using one incident point source wave is an open problem. For numerical inversion in the time-domain, Lines and Chandler-Wilde [18] have explored a time domain point source method and Burkard and Potthast [19] have developed a time domain probe method, based on the singular point source method of Potthast and the probe method of Ikehata et al. for bounded obstacle scattering problems in the frequency-domain, respectively. An alternative algorithm for (IP) is presented in [20], following the Kirsch–Kress optimization scheme developed first for acoustic obstacle scattering.

Next, we establish a uniqueness theorem with a countable number of incident point source waves, extending the idea of Kirsch and Kress [14] for bounded obstacles and that of Kirsch [2] for periodic structures to rough surface scattering problems.

Theorem 2.1 *The near-field data* $\{u(x; z_m) : x \in \Gamma_b^*\}$ *corresponding to a countable number of incident point source waves* $u^{in}(x; z_m)$ *with* $z_m \in \Gamma_c^*$, $m = 1,2,\ldots,$ *can determine the rough surface* Λ *uniquely. Here,* Γ_c^* *is another line segment above* Λ *satisfying* $\Gamma_b^* \cap \Gamma_c^* = \emptyset$.

Proof Let $\tilde{\Lambda}$ be another rough surface lying below Γ_b and Γ_c, and denote by $\tilde{u}(x; z)$, $\tilde{u}^{sc}(x; z)$ the total and scattered fields corresponding to the incident field $u^{in}(x; z)$ and $\tilde{\Lambda}$, and denote by $\tilde{D}$ the region above $\tilde{\Lambda}$. Assuming that

$$u(x; z_m) = \tilde{u}(x; z_m) \quad \text{for all } x \in \Gamma_b^*,\ x \neq z_m\ m = 1, 2, \ldots, \tag{5}$$

we shall prove $\Lambda = \tilde{\Lambda}$ by contradiction.

We first claim that $u(x; z) = \tilde{u}(x; z)$ for all $x \neq z$, $x, z \in \Omega$, where Ω denotes the unbounded connected component of $D \cap \tilde{D}$. Since u and $\tilde{u}$ are both analytic functions in Ω and $\Gamma_b \subset \Omega$, the identity (5) holds true for all $x \in \Gamma_b$. It follows from the uniqueness of the forward Dirichlet scattering problem over the half-space U_b that the identity (5) remains valid for all $x \in U_b$, and from the unique continuation of solutions to the Helmholtz equation that

$$u(x; z_m) = \tilde{u}(x; z_m) \quad \text{for all } x \in \Omega,\ x \neq z_m\ m = 1, 2, \ldots. \tag{6}$$

Recall [21, Theorem 3.1.4] that the solution $u(x; z)$ fulfils the reciprocity relation $u(x; z) = u(z; x)$ for all $x, z \in D$, $x \neq z$, and analogously that $\tilde{u}(x; z) = \tilde{u}(z; x)$ for all $x, z \in \tilde{D}$, $x \neq z$. Hence, by (6) we see that $u(z_m; x) = \tilde{u}(z_m; x)$ for all $x \in \Omega$, $m \in \mathbb{N}$. Setting $w(z) := u(z; x) - \tilde{u}(z; x)$ for some fixed $x \in \Omega$, we may conclude that w is analytic on Γ_c, with infinitely many zeros at $z = z_m$, $m \in \mathbb{N}$ on the finite line segment Γ_c^*. This implies that $w(z) = 0$ for all $z \in \Gamma_c$, i.e. $u(z; x) = \tilde{u}(z; x)$ for all $x \in \Omega$, $z \in \Gamma_c$, $x \neq z$. Repeating the arguments used in the proof of (6), we finally obtain the relation $u(x; z) = \tilde{u}(x; z)$ for all $x, z \in \Omega$, $x \neq z$. Since the scattered fields are continuous up to the boundary, there holds

$$u^{sc}(x; z) = \tilde{u}^{sc}(x; z) \quad \text{for all } x, z \in \overline{\Omega}. \tag{7}$$

If $\Lambda \neq \tilde{\Lambda}$, without loss of generality, we may assume that there exists $y_0 \in \Lambda \cap \tilde{D} \cap \partial\Omega$. Define a sequence $y_n := y_0 + (1/n)\mathbf{n}(y_0)$, $n \in \mathbb{N}$, such that $y_n \in \Omega$ for all sufficiently large $n \in \mathbb{N}$, where $\mathbf{n}(y_0)$ denotes the unit normal to Λ at y_0 pointing into D. On one hand, it follows from the smoothness of $\tilde{u}^{sc}(x; y_0)$ in $\tilde{D}$ that

$$\lim_{n \to +\infty} |\tilde{u}^{sc}(y_n; y_0)| = |\tilde{u}^{sc}(y_0; y_0)| < +\infty. \tag{8}$$

On the other hand, recalling the Dirichlet boundary condition $u^{in}(y_0; y_n) + u^{sc}(y_0; y_n) = 0$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to+\infty} |u^{sc}(y_n; y_0)| = \lim_{n\to+\infty} |u^{sc}(y_0; y_n)| = \lim_{n\to+\infty} |u^{in}(y_0; y_n)| = +\infty,$$

which contradicts (7) and (8). Thus $\Lambda = \tilde{\Lambda}$. ∎

Remark 2 The above approach does not depend on the kind of boundary conditions on Λ, but requires infinitely many incident point source waves. If the Dirichlet boundary condition is replaced with the impedance boundary condition, Theorem 2.1 still holds true; note that the well-posedness of (DP) under the impedance boundary condition is established in [22, Chapter 3] using the variational method.

2.2. *Uniqueness for polygonal periodic structures*

If rough surfaces are confined to periodic structures, the problem (DP) is always referred to as the grating diffraction problem. In this case, we can prove uniqueness to (IP) within polygonal periodic structures using only a single incident point source wave.

Assume that Λ is given by the graph of some 2π-periodic continuous piecewise linear function $x_2 = f(x_1)$, $x_1 \in \mathbb{R}$. A function $u : \mathbb{R}^2 \to \mathbb{C}$ is said to be quasi-periodic in x_1 with the phase-shift $\alpha \in \mathbb{R}$ if $u(x)\exp(-i\alpha x_1)$ is 2π-periodic with respect to x_1, or equivalently,

$$u(x_1 + 2\pi, x_2) = u(x_1, x_2)\exp(i2\pi\alpha), \quad x_1 \in \mathbb{R}.$$

An α-quasiperiodic incident wave $u^{in}(x; y)$ due to the point source $y \in D$ is defined by

$$u^{in}(x; y) = \sum_{n\in\mathbb{Z}} \frac{i}{4\pi\beta_n} e^{i[\alpha_n(x_1-y_1)+\beta_n|x_2-y_2|]}, \tag{9}$$

where $\alpha_n = n + \alpha$ and

$$\beta_n := \begin{cases} (k^2 - \alpha_n^2)^{\frac{1}{2}} & \text{if } |\alpha_n| \le k, \\ i(\alpha_n^2 - k^2)^{\frac{1}{2}} & \text{if } |\alpha_n| > k, \end{cases} \quad \text{with } i = \sqrt{-1}.$$

We assume that $\beta_n \neq 0$ for all $n \in \mathbb{Z}$, i.e. the Rayleigh frequencies are excluded. If the incident wave u^{in} is α-quasiperiodic in x_1 and $f(x_1)$ is periodic, it is proved in [9] that the scattered field must be also α-quasiperiodic and is uniquely solvable for either a plane wave incidence or a point source wave incidence. Under the α-quasiperiodicity assumption on u^{sc}, Chandler-Wilde [6] has shown that the UPRC can be rewritten more explicitly as the well-known Rayleigh expansion of the form

$$u^{sc} = \sum_{n\in\mathbb{Z}} A_n \exp(i\alpha_n x_1 + i\beta_n x_2) \quad \text{for } x_2 > f_+ := \max_{x_1\in\mathbb{R}} f(x_1), \tag{10}$$

where $A_n \in \mathbb{C}$ are called the Rayleigh coefficients. The well-posedness of the plane wave scattering in the diffraction grating case is proved by Kirsch [23] for a C^2-smooth boundary ∂D, by Elschner and Yamamoto [24] for a Lipschitz boundary ∂D, and by Elschner and Chandler-Wilde [9] for more general domains D fulfilling the condition

$$x = (x_1, x_2) \in D \Longrightarrow (x_1, x_2 + s) \in D \quad \text{for all } s > 0.$$

Our main result on uniqueness in determining polygonal periodic structures is as follows.

THEOREM 2.2 *A polygonal periodic structure* Λ *can be uniquely determined from the near-field data* $\{u(x; y)\colon 0 < x_1 < 2\pi,\ x_2 = b\}$, $b > f_+$, *corresponding to one incident point source wave* $u^{in}(x; y)$ *with* $y = (y_1, y_2)$ $y_2 > f_+$, $y_2 \neq b$.

Let $\tilde{\Lambda}$ be another 2π-periodic polygonal graph given by some function $\tilde{f}$ satisfying $y_2, b > \tilde{f}_+$, and let $\tilde{u}$, $\tilde{u}^{sc}$, $\tilde{D}$ and Ω be given as in Section 2.1. We need to prove that the relation

$$u(x_1, b; y) = \tilde{u}(x_1, b; y) \quad x_1 \in (0, 2\pi) \tag{11}$$

implies that $\Lambda = \tilde{\Lambda}$.

Definition 2.3 A straight line $l \subset D$ starting from one point and leading to infinity in $\{x_2 > b\}$ is called a Dirichlet ray of u if $u|_l = 0$.

From the identity (11), one can easily see that $u(x; y) = \tilde{u}(x; y)$ for all $x \in \Omega \backslash \{y\}$. According to the standard elliptic regularity theory, the total field $u(x; y)$ ($\tilde{u}(x; y)$) is infinitely smooth up to Λ ($\tilde{\Lambda}$) except for the corner points, and is analytic in $\Omega \backslash \{y\}$. Relying on the analyticity and the fact that both Λ and $\tilde{\Lambda}$ are piecewise linear, one can verify the following lemma.

LEMMA 2.4

(i) *If the relation* (11) *holds and* $\Lambda \neq \tilde{\Lambda}$, *then there always exists a Dirichlet ray* $l \subset \Omega$ *of both* u *and* $\tilde{u}$ *such that* l *is not parallel to the coordinate axes.*

(ii) *For non-periodic polygonal graphs* Λ *and* $\tilde{\Lambda}$, *the first assertion still holds true under one of the following additional assumptions*

(A1) *For each angle* ϕ *formed by the* x_1*-axis and a line segment of* $\Lambda \cup \tilde{\Lambda}$ *not parallel to the* x_1*-axis, we have* $|\tan(\phi)| > \epsilon$ *for some positive constant* ϵ.
(A2) *For each angle* ϕ *formed by the* x_1*-axis and a line segment of* $\Lambda \cup \tilde{\Lambda}$, *we have* $|\tan(\phi)| < M$ *for some* $M > 0$.

The key tool for proving Lemma 2.4 is the reflection principle for the Helmholtz equation under the Dirichlet boundary condition [15,25–27], which has been used to investigate uniqueness in inverse scattering by polygonal or polyhedral bounded obstacles with a single incident plane wave [25,26]. The reflection principle is stated as follows and will also be used in Section 2.3.

LEMMA 2.5 (Reflection principle) *Assume that* $\Omega \subset \mathbb{R}^2$ *is a symmetric domain with respect to the line* l, *and that* u *satisfies the Helmholtz equation* $(\Delta + k^2)\, u = 0$ *in* Ω *with* $u = 0$ *on* l. *Then* $u(x) = -u(\mathrm{Ref}_l(x))$ *in* Ω, *where* $\mathrm{Ref}_l(\cdot)$ *denotes the reflection with respect to the line* l. *In particular, if* $l' \subset \Omega$ *is another line* (*or line segment*) *such that* $u|_{l'} = 0$, *then* u *also vanishes on* $\mathrm{Ref}_l(l') \cap \Omega$.

We refer to [15, Lemma 2] for a detailed proof of Lemma 2.4(i) in periodic case, where the existence of the positive lower bound in (A1) is always guaranteed.

With necessary modifications, the proof can be readily carried over to general non-periodic polygonal structures which fulfil condition (A2). Note that (A2) implies that both Λ and $\tilde{\Lambda}$ are given by piecewise linear functions with uniformly bounded Lipschitz constants. Based on Lemma 2.4 and the reduction argument in [15, Lemma 3], we next prove Theorem 2.2 using a single incident point source wave.

Proof of Theorem 2.2 We begin with decomposing the incident point source wave $u^{in}(x; y)$ into upward modes and downward modes by

$$u^{in}(x;y) = \begin{cases} \sum_{n\in\mathbb{Z}} B_n^+(y)\exp(i(\alpha_n x_1 + \beta_n x_2)) & \text{in } x_2 \geq y_2, \\ \sum_{n\in\mathbb{Z}} B_n^-(y)\exp(i(\alpha_n x_1 - \beta_n x_2)) & \text{in } x_2 < y_2, \end{cases} \qquad x \neq y,$$

where $B_n^{\pm}(y) := \exp(i(-\alpha_n y_1 \mp \beta_n y_2))i/(4\pi\beta_n)$. If (11) holds but $\Lambda \neq \tilde{\Lambda}$, by Lemma 2.4 we may assume without loss of generality that there exists a Dirichlet ray $l := \{(t, at) : t > 0\}$ for some $a > 0$. Hence, for $t \in T := \{t > 0 : at > y_2\}$, there holds

$$\begin{aligned} 0 = U(t) &:= u(t, at; y) = u^{in}(t, at; y) + u^{sc}(t, at; y) \\ &= \sum_{n\in\mathbb{Z}} B_n^+(y)\exp(i(\alpha_n t + \beta_n at)) + \sum_{n\in\mathbb{Z}} A_n(y)\exp(i(\alpha_n t + \beta_n at)) \\ &= \sum_{|\alpha_n|\leq k} (B_n^+(y) + A_n(y))\exp(i(\alpha_n t + \beta_n at)) \\ &\quad + \sum_{|\alpha_n|>k} (B_n^+(y) + A_n(y))\exp(i\alpha_n t - |\beta_n|at) \\ &=: V(t) + W(t). \end{aligned} \tag{12}$$

One can observe that $W(t)$ consists of exponentially decaying functions as $t \to +\infty$. Thus, for any $\epsilon > 0$, there exists $t_0 \in T$ sufficiently large such that $|W(t)| < \epsilon$ for all $t > t_0$. Together with (12), this leads to $|V(t)| < \epsilon$ for $t > t_0$. However, since $V(t)$ is an almost periodic function on $\mathbb{R}$, it holds that

$$\max_{t\in\mathbb{R}} |V(t)| = \limsup_{t\to+\infty} |V(t)| < \epsilon.$$

Thus, by the arbitrariness of ϵ, we arrive at $V(t) = 0$ for all $t \in \mathbb{R}$, which implies that $W(t) = 0$ for all $t \in T$. Now, using the argument employed in [15, Lemma 3], we can conclude that $B_n^+(y) + A_n(y) = 0$ for $|\alpha_n| > k$. Therefore, the total field can be reduced to a finite number of propagating modes

$$u(x;y) = u^{in}(x;y) + u^{sc}(x;y) = \sum_{|\alpha_n|\leq k} (B_n^+(y) + A_n(y))\exp(i(\alpha_n x_1 + \beta_n x_2))$$

in $x_2 > y_2$, which is an analytic function in the region $x_2 > y_2$. Moreover, the solution $u(x; y)$ remains bounded as x tends to y in the half-space U_{y_2}. However, since $u^{sc}(x; y)$ is smooth in a neighbourhood of y and $u^{in}(x; y)$ has the same singularity as the free-space fundamental solution of the two-dimensional Helmholtz equation [23], the limit of $u(x; y)$ as $x \to y$ must be unbounded. This contradiction implies that $\Lambda = \tilde{\Lambda}$. ■

Remark 3 Theorem 2.2 remains valid for the Neumann boundary condition. Analogously, one can prove that one incident quasi-periodic point source wave is sufficient to determine a bi-periodic polyhedral grating profile under the perfect conductor boundary condition (the tangential components of electric field vanish) and under the third or fourth kind boundary conditions of linear elasticity. Note that in all these cases, one incident plane wave is not enough in general to determine a grating profile uniquely; see [13,15,28,29]. However, such a reduction argument relies heavily on the Rayleigh expansion of the scattered fields, and it seems impossible to extend this argument to non-periodic polygonal structures where the UPRC is used.

In the next section, we adopt another approach to prove uniqueness for rough polygonal surfaces, providing a new proof of Theorem 2.2.

2.3. *Uniqueness for non-periodic polygonal surfaces*

THEOREM 2.6 *Let Γ_c^* $(c>0)$ and Γ_b^* $(b>0)$ be two different line segments parallel to the x_1-axis satisfying $\Gamma_b^* \cap \Gamma_c^* = \emptyset$, and define the incident point source waves $u^{in}(x;y)$ for some $y \in \Gamma_c^*$ as in* (3). *Suppose that the scattering surface Λ is the graph given by some continuous piecewise linear function $f(x_1)$, satisfying $|f(x_1)| < \min\{b, c\}$ for all $x_1 \in \mathbb{R}$ and one of the conditions* (A1) *and* (A2) *in Lemma* 2.5(ii). *Then, the near-field data $\{u(x;y) : x \in \Gamma_b^*\}$ determine the rough surface Λ uniquely.*

Proof Assume that $\tilde{\Lambda}$ is another one-dimensional scattering surface satisfying all the conditions imposed on Λ in Theorem 2.6. Denote by $\tilde{u}(x;y)$ the total field corresponding to u^{in} and $\tilde{\Lambda}$. If $u(x;y) = \tilde{u}(x;y)$ on Γ_b^*, then similarly to the proof of (6), one arrives at $u(x;y) = \tilde{u}(x;y)$ for all $x \in \Omega \backslash \{y\}$, where Ω denotes again the unbounded connected component of $D \cap \tilde{D}$.

Assume that $\Lambda \neq \tilde{\Lambda}$. It follows from Lemma 2.4(ii) that there exists at least one Dirichlet ray l which is not parallel to the coordinate axes. Since $u = u^{in} + u^{sc}$ vanishes on l and the incident field is singular at $x = y$, we see that l cannot pass through the point source y. If y lies below the Dirichlet ray l, then the point $\mathrm{Ref}_l(y)$ must lie above l, which implies that $\mathrm{Ref}_l(y) \in U_c := \{x_2 > c\} \subset \Omega$. Then, applying the reflection principle of Lemma 2.5 yields the relation $u(x;y)|_{x=y} = -u(x;y)|_{x=\mathrm{Ref}_l(y)}$, which is impossible since $u(x;y)$ is singular at $x = y$, while $u(x;y)$ remains bounded as $x \to \mathrm{Ref}_l(y)$. Thus it remains to consider the case when y lies above l, where the point $\mathrm{Ref}_l(y)$ may lie in $\mathbb{R}^2 \backslash \Omega$. However, we claim that in this case there exists another ray $l' \subset \Omega$ such that $u|_{l'} = 0$ and $\mathrm{Ref}_{l'}(y) \in \Omega$, which would analogously lead to the same contradiction.

Without loss of generality we denote by $l = \{(t, at) : t > 0\}$ for some $a > 0$. The case $a < 0$ can be treated similarly. Since $\partial\Omega$ is the graph of some continuous piecewise linear function, we can always choose two neighbouring line segments $A_1O, OA_2 \in \partial\Omega$ with the end points $A_1, O, A_2 \in \mathbb{R}^2$ such that

$$\min\{x_1 : x = (x_1, x_2) \in A_1O \cup OA_2\} > n$$

for some $n > 0$. Since $\partial\Omega$ is bounded in the x_2-direction, the Huasdorff distance between $A_1O \cup OA_2$ and l tends to infinity as $n \to +\infty$. Thus we have $\mathrm{Ref}_l(A_1O \cup OA_2) \subset U_c$ provided $n > 0$ is sufficiently large. Then we extend the line segments $\mathrm{Ref}_l(A_1O)$, $\mathrm{Ref}_l(OA_2)$ to the rays l_1', l_2' in Ω, respectively. By the reflection principle, we see that u vanishes on both l_1' and l_2', and thus also vanishes on an equi-angular system Σ formed by l_1' and l_2', where the set $\Sigma \subset \Omega$ consists of a finite number of rays passing through the same point $O' := l_1' \cap l_2' \in U_c$; see [13, Lemma 6]. Now, we see that there exists at least one ray $l' \subset \Sigma$ such that $u|_{l'} = 0$ and $\mathrm{Ref}_{l'}(y) \in \Omega$. The proof is thus complete. ■

3. Uniqueness for the transmission problem

In this section, we consider the scattering of a time-harmonic electromagnetic wave by several isotropic rough layers. Suppose that the medium varies only in x_1-direction and is constant in x_3-direction. We restrict ourselves to the case of two rough interfaces, and consider the TM mode (transverse magnetic polarization) where the time-harmonic Maxwell equations can be reduced to a two-dimensional scalar Helmholtz equation with the TM transmission conditions imposed on each rough interface.

Let the cross-sections Λ_j of the rough interfaces in the (x_1, x_2)-plane be given by graphs of disjoint $C^{1,1}$ functions $\Lambda_j := \{x_2 = f_j(x_1), x_1 \in \mathbb{R}\}$, $j = 1,2$, satisfying

$$f_1(\tilde{x}) > f_2(\tilde{x}), \quad |f_j(\tilde{x}) - f_j(\tilde{y})| \le L_j\,|\tilde{x} - \tilde{y}|, \quad \text{for all } \tilde{x}, \tilde{y} \in \mathbb{R}^{n-1}, \tag{13}$$

with $L_j > 0$, $j = 1, 2$. Denote the region above Λ_1 by D_0, the one below Λ_2 by D_2, and that between Λ_1 and Λ_2 by D_1; see Figure 1. The three distinct constant refractive indices corresponding to D_j are denoted by k_j ($i = 0,1,2$), respectively, satisfying $k_j > 0$, and $k_0 \ne k_1$, $k_1 \ne k_2$. Let

$$\Lambda_1^+ := \max_{x_1 \in \mathbb{R}}\{f_1(x_1)\}, \quad \Lambda_2^- := \min_{x_1 \in \mathbb{R}}\{f_2(x_1)\}.$$

Suppose that from the top region D_0 we have an incident wave $u^{in}(x; y)$ due to the point source $y \in D_0$ defined by (3) with k replaced by k_0. Then, the total field $u = u(x; y)$ satisfies

$$\Delta u + k_j^2 u = 0 \quad \text{in } D_j \backslash \{y\}, j = 0, 1, 2, \tag{14}$$

$$u^+ = u^-, \quad \frac{1}{k_{j-1}^2}\frac{\partial u^+}{\partial \mathbf{n}} = \frac{1}{k_j^2}\frac{\partial u^-}{\partial \mathbf{n}} \quad \text{on } \Lambda_j,\ j = 1, 2, \tag{15}$$

$$u = u^{in}(x; y) + u^{sc}(x; y) \quad \text{in } D_0, \tag{16}$$

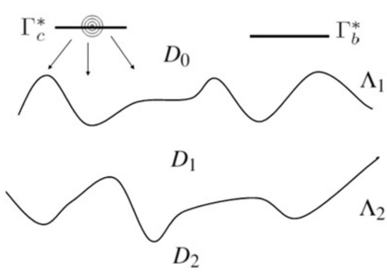

Figure 1. The geometric figure of the background medium.

where $\mathbf{n}$ denotes the unit normal to Λ_j pointing into D_{j-1}, and u^+, $\frac{\partial u^+}{\partial \nu}$ (resp. u^-, $\frac{\partial u^-}{\partial \nu}$) denote the limits of u on Λ_j from above (resp. below). The scattered field u^{sc} is required to satisfy the UPRC (4) in D_0 for some $H > \Lambda_1^+$ with k replaced with k_0, while the field u in D_2 is required to satisfy the downward propagating radiation condition (DPRC):

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(i[-(x_2-h)\sqrt{k_2^2-\xi^2}+x_1\xi])\hat{F}_h(\xi)\mathrm{d}\xi, \quad x\in\mathbb{R}^2\backslash\overline{U}_h, \tag{17}$$

where $F_h := u|_{\Gamma_h} \in L^2(\Gamma_h)$ for some $h < \Lambda_2^-$, and $\sqrt{k_2^2-\xi^2} = i\sqrt{\xi^2-k_2^2}$ when $|\xi|>k_2$.

For $y\in\mathbb{R}^2\backslash(\Lambda_1\cup\Lambda_2)$, the function $G(x;y)$ is called the fundamental solution to the above scattering problem if there holds

$$\left.\begin{array}{c} L_xG(x;y) := \nabla\cdot(a\nabla G(x;y)) + G(x;y) = -\delta(x-y), \quad \text{in } \mathbb{R}^2, \\ G^+ = G^-, \quad a^+\frac{\partial G^+}{\partial \mathbf{n}} = a^-\frac{\partial G^-}{\partial \mathbf{n}}, \quad \text{on } \Lambda_j,\ j=1,2, \\ G(x;y) \text{ satisfies the UPRC (4) with } k=k_0 \text{ and the DPRC (17) ,} \end{array}\right\} \tag{18}$$

where $a(x) = 1/(k_j^2)$ for $x\in D_j$, $j=0,1,2$. One can further observe that the fundamental solution $G(x;y)$ coincides with the function $k_0^2u(x;y)$ if the point source $y\in D_0$. We next prove that the fundamental solution exists and is unique under some monotonicity conditions imposed on k_j, from which the well-posedness of our transmission problem (14)–(17) also follows. We assume that, for $y\notin\Lambda_1\cup\Lambda_2$, the function

$$x\mapsto(1-\chi(\|x-y\|\epsilon^{-1}))G(x;y)$$

belongs to $H^1(U_h\backslash\overline{U_H})$ for each $\epsilon>0$. Here $\chi(t)$ is a smooth function on $[0,+\infty)$ satisfying $\chi(t)=1$ for $t\le 1/2$ and $\chi(t)=0$ for $t\ge 1$.

LEMMA 3.1 *For $y\notin\Lambda_1\cup\Lambda_2$, the Green function $G(x;y)$ exists and is unique if one of the following conditions is satisfied:*

$$\text{(i)}\ \ k_0>k_1>k_2; \qquad \text{(ii)}\ \ k_0>k_1, k_1<k_2; \qquad \text{(iii)}\ \ k_0<k_1<k_2. \tag{19}$$

Proof Without loss of generality, we may assume that $y\in D_N$ for some $N\in\{0,1,2\}$ is a fixed point source. Let $\eta>0$ denote the Hausdorff distance between y and $\Lambda_1\cup\Lambda_2$, and choose a smooth function $\tilde{\chi}(t)\in C^\infty(\mathbb{R}^+)$ satisfying $\tilde{\chi}(t)=1$ for $t<\eta/4$, and $\tilde{\chi}(t)=0$ for $t>\eta/2$. Setting $V(x;y)=G(x;y)-U(x;y)$, where $U(x;y) := (i/4)H_0^{(1)}(k_N|x-y|)\chi(|x-y|)k_N^2$, we see that

$$\Delta_x V(x;y) + k_j^2V(x;y) = g, \quad \text{in } D_j,\ j=0,1,2,$$

where $g(x)$ is some C^∞ smooth function on $\mathbb{R}^2$ compactly supported in D_N. Moreover, $V(x)$ satisfies the TM transmission conditions (18) on Λ_j, the UPRC (4) with $k=k_0$ and the DPRC (17). Thus, under one of the conditions in (19) it follows from [30, Corollary 2.3] that $V(x)\in H^1(U_h\backslash\overline{U}_H)$ is the unique solution to this transmission problem and satisfies the estimate $\|V\|_{H^1(U_h\backslash\overline{U}_H)}\le C\|g\|_{L^2(D_N)}$ for some constant $C>0$ depending on H, h, k_j $(j=0,1,2)$ and Λ_j $(j=1,2)$. Thus, $G(x;y)=V(x;y)+U(x;y)$ is the unique fundamental solution to the transmission problem (14)–(17). ∎

We remark that the Green function $G(x;y)$ satisfies the reciprocity relation $G(x;y)=G(y;x)$ for all $x,y \notin \Lambda_1 \cup \Lambda_2, x \neq y$. This property simply follows from the fact that the variational formulation corresponding to (18) is symmetric, since the leading coefficient $a(x)$ of the differential equation in (18) is real and positive definite. One can also prove this using Green's formula combined with the transmission conditions on Λ_j and the radiation conditions UPRC and DPRC; we refer to [21, Theorem 3.1.4] for the proof in a half plane under the Dirichlet boundary condition.

Our inverse problem in this section is as follows.

(IP′) Given $k_0>0$ and the infinitely many incident waves $u^{in}(x;z_m)$ generated by point sources $z_m \in \Gamma_c^*$ $(m=1,2,\ldots)$, determine the rough interfaces Λ_j and the refractive indices k_j $(j=1,2)$ from the knowledge of the near-field data $\{u(x;z_m): x \in \Gamma_b^*, m \in \mathbb{N}\}$, where $\Gamma_b^* \cap \Gamma_c^* = \emptyset$.

We next extend the arguments from Section 2.1 to prove uniqueness in (IP′).

Theorem 3.2 *Under one of the conditions in* (19), *the rough interfaces* Λ_j *with* $j=1,2$, *and the constant refractive indices* k_j, $j=1,2$, *can be uniquely determined from the near-field data* $\{u(x;z_m): x \in \Gamma_b^*, m \in \mathbb{N}\}$ *corresponding to the infinitely many incident point source waves* $u^{in}(x;z_m)$, $m \in \mathbb{N}$.

Note that the approach of Kirsch and Kress for proving uniqueness using point sources has already applied to penetrable obstacles (see, e.g. [5,14]). In this article, we adopt the idea from [16] of the leading singularity of the fundamental solution $G(x;y)$, which allows us to prove Theorem 3.2 in a more straightforward way. This section also extends the results of [16] to the case of non-periodic unbounded penetrable obstacles.

Given two functions $f(x)$ and $g(x)$, we say that $f(x)\sim g(x)$ as $x \to x_0$ if $\lim_{x\to x_0} f(x)/g(x)=1$. Obviously, if $f(x)$, $g(x)\to\infty$ as $x\to x_0$ and $f(x)-g(x)$ is bounded in a neighbourhood of x_0, then $f(x)\sim g(x)$ as $x\to x_0$. Analogously, given two sequences f_n and g_n, we say that $f_n \sim g_n$ as $n\to+\infty$ if $\lim_{n\to\infty} f_n/g_n=1$. If $y_0 \in D_j$ for some $j\in\{0,1,2\}$, it can be readily deduced from the fundamental solution to the two-dimensional Laplace equation that

$$G(x;y_0) \sim -\frac{k_j^2}{2\pi}\ln\|x-y_0\| \quad \text{as } x \to y_0. \tag{20}$$

Note that the relation (20) only depends on the wave numbers k_j corresponding to D_j. However, we do not know the existence of the Green function $G(x;y)$ in the case that y belongs to the interfaces Λ_j $(j=1,2)$. Given $y_0 \in \Lambda_j$, $j\in\{1,2\}$, define a sequence y_n by

$$y_n = y_0 + \frac{1}{n}\mathbf{n}(y_0), \quad n=1,2,\ldots. \tag{21}$$

The reciprocity relation for $G(x;y)$ allows us to define $G(y_n;y_0)$ for fixed n by

$$G(y_n;y_0) := G(y_0;y_n) = \lim_{m\to+\infty} G\left(y_0+\frac{1}{m}\mathbf{n}(y_0);y_n\right);$$

note that the limit exists because Λ_j is $C^{1,1}$-smooth and the function $G(\cdot;y_n)$ is continuous up to Λ_j. We recall the following lemma on the limit of $G(y_n;y_0)$ for

$y_0 \in \Lambda_1 \cup \Lambda_2$ as $n \to +\infty$, which is proved in [16, Lemma 2.5] by employing Fourier transform under the condition that Λ_j are C^2-smooth. The result remains valid if Λ_j are given by $C^{1,1}$-smooth functions.

LEMMA 3.3 *For fixed* $y_0 \in \Lambda_j$, $j \in \{1,2\}$, *we have*

$$G(y_n; y_0) \sim -\frac{k_j^2 k_{j-1}^2}{\pi(k_{j-1}^2 + k_j^2)} \ln \| y_n - y_0 \| \quad \text{as } n \to +\infty,$$

where the sequence y_n *is defined by* (21).

Now, based on Lemma 3.3 and the relation (20) we sketch the proof of Theorem 3.2, following the steps in the proof of [16, Theorem 2.1] for multilayered bounded obstacles.

Proof of Theorem 3.2 Let $\tilde{\Lambda}_j$ $(j=1,2)$ be another two disjoint rough interfaces separating the regions $\tilde{D}_j$ $(j=0,1,2)$, with the wave number $\tilde{k}_j$ in $\tilde{D}_j$ $(j=1,2)$ satisfying $k_0 \neq \tilde{k}_1$, $\tilde{k}_1 \neq \tilde{k}_2$. Analogously, we use $\tilde{u}$, $\tilde{u}^{sc}$ and $\tilde{G}(x;y)$ to denote the corresponding fields and fundamental solution related to the rough layers characterized by $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ and $\tilde{k}_1, \tilde{k}_2$. Supposing that the identity (5) holds, we shall prove that $\Lambda_j = \tilde{\Lambda}_j$ and $k_j = \tilde{k}_j$ for $j=1,2$.

Assume that $\Lambda_1 \neq \tilde{\Lambda}_1$. Without loss of generality, we may assume that there exists $y_0 \in \tilde{\Lambda}_1 \cap D_0 \cap \partial\Omega$, where Ω denotes the unbounded connected component of $D_0 \cap \tilde{D}_0$. Let y_n be defined as in (21), and let the functions $F(x), \tilde{F}(x)$ be given by

$$F(x) := -2\pi G(x; y_0)/\ln \|x - y_0\|, \quad \tilde{F}(x) := -2\pi \tilde{G}(x; y_0)/\ln \|x - y_0\|. \tag{22}$$

Since $y_n \in D_0 \cap \Omega$ for sufficiently large n, it follows from (20) and Lemma 3.3 with $\Lambda_j = \Lambda_1$ that

$$\lim_{n\to+\infty} F(y_n) = k_0^2, \quad \lim_{n\to+\infty} \tilde{F}(y_n) = 2k_0^2\tilde{k}_1^2/(k_0^2 + \tilde{k}_1^2),$$

leading to

$$\lim_{n\to+\infty} [F(y_n) - \tilde{F}(y_n)] = k_0^2(k_0^2 - \tilde{k}_1^2)/(k_0^2 + \tilde{k}_1^2). \tag{23}$$

However, using the same argument as in the proof of Theorem 2.1, one can derive from the equality (5) that $G(x;y) = \tilde{G}(x;y)$ for all $x, y \in \overline{\Omega}$, and thus that $\tilde{F}(y_n) = F(y_n)$ for all sufficiently large $n \in \mathbb{N}$, which contradicts (23) because $k_0 \neq \tilde{k}_1$. Hence $\Lambda_1 = \tilde{\Lambda}_1$.

We next prove that $k_1 = \tilde{k}_1$. Choose $y_0 \in \Lambda_1 = \tilde{\Lambda}_1$, and define y_n, $F(x)$, $\tilde{F}(x)$ in the same way as in (21) and (22). Applying Lemma 3.3 again yields the identity

$$0 = \lim_{n\to+\infty} [F(y_n) - \tilde{F}(y_n)] = \frac{2k_0^2k_1^2}{k_0^2 + k_1^2} - \frac{2k_0^2\tilde{k}_1^2}{k_0^2 + \tilde{k}_1^2} = \frac{2k_0^4(k_1^2 - \tilde{k}_1^2)}{(k_0^2 + k_1^2)(k_0^2 + \tilde{k}_1^2)},$$

from which $k_1 = \tilde{k}_1$ follows.

Finally, applying Holmgren's uniqueness theorem gives $G(x;y) = \tilde{G}(x;y)$ for all $x \neq y, x, y \in \overline{\Omega}_0$, where Ω_0 denotes the unbounded connected component of $(\mathbb{R}^2 \backslash \overline{D}_2) \cap (\mathbb{R}^2 \backslash \overline{\tilde{D}}_2)$. Proceeding in a similar way, we can prove that $\Lambda_2 = \tilde{\Lambda}_2$ and $k_2 = \tilde{k}_2$. ∎

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