# In-plane ferromagnetic instability in a two-dimensional electron liquid in the presence of Rashba spin-orbit coupling

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**Abstract.** We show that due to the peculiar structure of the non-interacting energy spectrum, the Coulomb interaction leads for all densities to an in-plane ferromagnetic instability in a two-dimensional electron liquid in the presence of sufficiently strong Rashba spin-orbit coupling. This non perturbative phenomenon is characterized by an interesting anisotropic momentum space repopulation and is in nature quite different from the already identified out-of-plane ferromagnetic instability.

The collection of articles in this volume will testify better than me the exuberant and passionate character of Gabriele, as well as the broad range of his interests and scientific contributions. Here I will reproduce (in the next section) an older unpublished manuscript, which originated from the research Prof. Giuliani and I carried out in Purdue in the period from 2002 until 2007, when Gabriele was my PhD advisor and an invaluable example for both scientific and human aspects. The imaginative title of the next Section is borrowed from one of his talks (see Figure 1). The pervasive humor of Gabriele is still at work today: I was reminded of it after searching online without success for the SCEM06 conference cited in his slides. In the spirit of this volume, I hope my introduction will further illustrate the unconventional personality and gifts of Gabriele. Besides mentioning a few memories from when I was a PhD student, I will review some of his scientific ideas from that period and try to connect them with more recent literature, which is obviously missing in the original manuscript.

As a student, as a matter of fact, I regularly approached his office with a feeling of uncertainty. First of all, the electric lights could be seen through the closed door but were generally turned on at any time of the day and night, making it difficult to guess if the office was occupied or not. The light shined through a semi-transparent glass mostly covered by old newspaper clippings (among which a large headline: *Could anyone be worse than Koch? Try Giuliani*). So the worries related to the ongoing research were slightly amplified at the door. After a few moments trying to detect any noise, I would hold on to my notes and knock. During the meetings I would be dragged in a whirlpool of ideas intermixed with a string of provocative remarks, anecdotes, and various considerations on physics and a wide range of other subjects often including soccer (of which Gabriele was a great lover). When I left the office, I was usually quite puzzled on the outcome of the discussion and what to do next. The views of Prof. Giuliani on our ongoing research seemed at first rather paradoxical or far-fetched to my cautious and inexperienced attitude, but they would reveal themselves in due time as useful and deeply true, such that my PhD turned out in the end to be a very productive period of research.


**Figure 1.** Gabriele loved to wrap physical concepts in colorful terms. In his talks, the topic of this article was introduced as a 'crescent moon' instability (by analogy to the left panel of Figure 4). Other noticeable slides from the same presentation (SCEN06, Pisa) are the *Four pere intermission* (featuring a short video of F. Totti) and *How do we do our calculations? Buy the book!* (obviously referring to [14]).

Some of the ideas he formulated in our discussions have shown in my opinion a remarkable foresight. For example, after we worked out the phase diagram in Figure 3 [1] he liked to mention that the formation of the thin sleeve of spin-polarized states along the dashed curve is very analogous to what happens in the Peierls instability [2]. The dahsed curve indicates when the Fermi energy is crossing what he called the 'kissing point' of the two spin bands (see the left panel of Figure 3). In this case, the spontaneous polarization arises by the formation of a gap which removes the degeneracy and leads to a lower total energy of the occupied spin branch. Interestingly, this picture is related to more recent studies of a spontaneous helical nuclear-electronic spin polarization in quantum wires [3]. The formation of these helical states can also be seen (after a gauge transformation generating a Rashba spin-orbit interaction of suitable strength) as due to a similar Peierls-type instability, where the coupled electron-nuclear spin polarization induces a finite gap at the $k = 0$ band crossing of the one-dimensional electron states [4]. The formation of the gap could be detected in transport and experimental evidence in this direction was recently reported [5].

Beside the FZ ferromagnetic phase shown in Figure 3, in-plane polarized states appear in the complete phase diagram in the Hartree-Fock approximation [6]. Gabriele liked to contrast the 'tilting instability' of spin directions, giving rise to the FZ polarized states, to the 'repopulation instability', a general mechanism giving rise to the in-plane ferromagnetic states. This type of instability is the main topic of the present article. Following his suggestion, the instability could also be studied at large spin-orbit coupling through linear response, from the divergence of the in-plane Pauli spin susceptibility [6]. The instability eventually gives rise to in-plane spin-polarized states with strongly deformed oblong or even 'bean-shaped' occupations, schematically illustrated in the right panel of Figure 4. Recently, the occurrence of these states was proposed in a variety of systems: bilayer graphene [7], electron liquids with short-range interactions [8], and spinor Bose gases [9].

Interestingly, the competition between the FZ phase and in-plane polarized states gives rise of another peculiar feature in the Hartree-Fock phase diagram. When the spin-orbit coupling $\alpha$ approaches zero ($\alpha = 0^+$), the boundary between PM and FZ still occurs at $r_s \simeq 2.01$ as in Figure 3 but a distinct phase boundary between FZ and the in-plane polarized states survives at $r_s = 2.211$ [6]. This behavior is peculiar because at $\alpha = 0$ the magnetic phase is fully isotropic (i.e., there is a single phase boundary at $r_s \simeq 2.01$, the well known Bloch transition [10]) and Gabriele liked particularly this curious 'non-analytic' phase transition in $\alpha, r_s$. Although in this case I am not able to point the reader to related literature, I would not be surprised if a similar phenomenon could play an important role in other contexts, given his perceptive intuition!

In a similar way as with physics, Gabriele was also a supportive advisor from the personal point of view. While utterly defiant of his disease, he was promptly ready to help us in case of difficulties with his characteristic decisiveness. His sometimes challenging attitude was always directed to stimulate students and co-workers to achieve the best outcomes. Also for this he will be deeply missed.

## 1. The crescent moon instability

The problem of a two-dimensional electron liquid in the presence of spin-orbit coupling of the Rashba type is not only of fundamental importance but also of particular technological relevance in view of the considerable recent interest in the possibility of manipulating electronic spins by electric means in modern devices [11–13].

While the effect of the electron interaction in the clean two-dimensional electron liquid is a classic problem studied now for decades [14], the intriguing interplay of many body effects and spin-orbit coupling has only recently began to receive serious attention. The simplest approaches, still not completely characterized, are the random phase approximation (RPA) and the Hartree-Fock (HF) theory. Within the RPA some of the approximate quasiparticle properties were studied in Ref. [15], while the corresponding diagrammatic expansion has been used to extract what amounts to as the exact behavior of the system at high densities in Ref. [16]. Albeit approximate, the HF mean field theory is at the moment the most promising framework to examine the phase diagram. In this respect the behavior of and the observable effects [17] related to the exchange energy in a quantum well were investigated in Ref. [18] for a generalized form of spin-orbit coupling. Furthermore the structure of the HF theory and the peculiar extension of the classic Bloch transition to a homogeneous polarized phase scenario [14] was examined in Ref. [19]. Finally the relevance in this problem of spatially inhomogeneous, charge and spin-density-wave distorted HF states was investigated in Ref. [20].

The purpose of the present paper is to point out the existence in this system of an interesting in-plane ferromagnetic instability of the paramagnetic state that occurs for sufficiently strong Rashba spin-orbit coupling for all densities. The phenomenon acquires particular interest since it is characterized by a peculiar breaking of the rotational symmetry of the momentum space occupation. In this respect it can be seen to be quite different from the already identified out-of-plane ferromagnetic transition. Although we will identify and prove the existence of this non perturbative behavior within the HF theory, the physics underlining the phenomenon is such that it is reasonable to expect that correlation effects will enhance it.

The non interacting problem is defined by the following single-electron Hamiltonian

$$\hat{H}_0 = \frac{\hat{\mathbf{p}}^2}{2m} + \alpha \left( \hat{\sigma}_x \hat{p}_y - \hat{\sigma}_y \hat{p}_x \right) , \tag{1.1}$$

that describes motion limited to $x$-$y$ plane and includes a (linear) spin-orbit interaction of the Rashba type [21,22], with $\alpha$ assumed to be positive. The corresponding single-particle eigenfunctions and eigenvalues are given by

$$\varphi_{\mathbf{k},\pm}(\mathbf{r}) = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{2L^2}} \begin{pmatrix} \pm 1 \\ i e^{i\phi_{\mathbf{k}}} \end{pmatrix} , \quad \epsilon_{\mathbf{k}\pm} = \frac{\hbar^2 \mathbf{k}^2}{2m} \mp \alpha\hbar\, k , \tag{1.2}$$

where $L$ is the linear size of the system and $\phi_{\mathbf{k}}$ is the angle between the direction of the wave vector and the $x$-axis. Plots of the non interacting spectrum are provided in Figure 2. It is important to notice that the locus of the points of minimum energy is a circle of radius given by $\frac{m\alpha}{\hbar}$, and that the spinor in $\varphi_{\mathbf{k},+(-)}(\mathbf{r})$ is parallel (antiparallel) to the unit vector $\hat{\phi}_{\mathbf{k}} = -\sin\phi_{\mathbf{k}}\hat{x} + \cos\phi_{\mathbf{k}}\hat{y}$, and therefore can be assigned a positive (negative) chirality.

The corresponding many-body problem is then obtained by accounting for the electronic Coulomb interaction and a suitable homogeneous and rigid neutralizing background [14]. While in the absence of the spin-orbit terms the relevance of the interaction is solely determined by the dimensionless density parameter $r_s^{-1} = \sqrt{\pi a_B^2 n}$ ($n$ being the electron density), here we must also include in our considerations a second dimensionless parameter, *i.e.* $\bar{\alpha} = \frac{\hbar\alpha}{e^2}$: the interplay of these two quantities is responsible for a rich physical scenario.

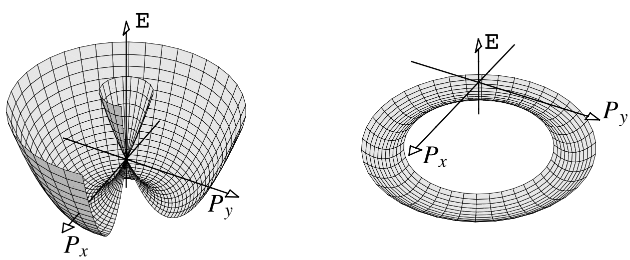


**Figure 2.** Non interacting single particle spectrum in the presence of linear Rashba spin-orbit interaction. Left: momentum space occupation for small spin-orbit coupling, or large density, with generalized chirality less than one. Right: case of large spin-orbit coupling, or small density, when only the bottom of the lowest band is occupied, a situation in which the generalized chirality is larger than one.

As described in Ref. [19], if one limits the analysis to spatially homogeneous states described by single Slater determinants of plane waves, one finds that the only relevant degree of freedom is the orientation $\hat{s}_{\mathbf{k}}$ of the spin quantization axis of each of the momentum states. As a consequence the HF energy will be in general a functional of $\hat{s}_{\mathbf{k}}$ and the occupation numbers $n_{\mathbf{k}\mu}$. This quantity is readily obtained and is given by:

$$\begin{aligned}\mathcal{E}[n_{\mathbf{k}\mu}, \hat{s}_{\mathbf{k}}] = & \sum_{\mathbf{k};\,\mu=\pm}\left(\frac{\hbar^2\mathbf{k}^2}{2m}\, n_{\mathbf{k}\mu} - \hbar\alpha\mu\, k\, \hat{\phi}_{\mathbf{k}}\cdot\hat{s}_{\mathbf{k}}\, n_{\mathbf{k}\mu}\right) \\ & -\frac{1}{4L^2}\sum_{\mathbf{k},\mathbf{k}';\,\mu,\mu'=\pm} v_{\mathbf{k}-\mathbf{k}'}\,(1+\mu\mu'\,\hat{s}_{\mathbf{k}}\cdot\hat{s}_{\mathbf{k}'})\, n_{\mathbf{k}\mu}n_{\mathbf{k}'\mu'}\,, \quad (1.3)\end{aligned}$$

where the first line describes the one particle terms, kinetic plus spin-orbit, and the second the exchange energy. For states constructed with symmetric occupation in momentum space the already established phase diagram is depicted in Figure 3 [19]. Neglecting low density phases that are tantamount to a magnetized Wigner crystal, one can identify a paramagnetic chiral phase (PM), that displays a reentrant behavior, and an out-of-plane ferromagnetic chiral phase (FZ) that can be seen as an extension to finite $\bar{\alpha}$ of the classic Bloch instability. Here the latter owes its existence at higher densities to the cusp characterizing the single particle spectrum (see Figure 2), and displays a non trivial spin texture in momentum space [19]. Notice that the FZ phase can persist in a ever shrink-

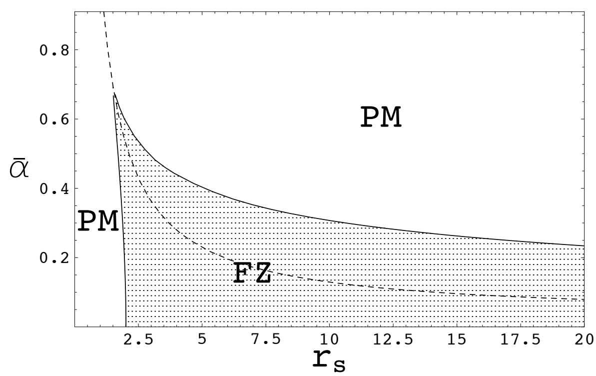


**Figure 3.** Mean field phase diagram limited to solutions with symmetric momentum space occupation. Within the shaded area the system finds itself in an out-of-plane ferromagnetic chiral phase (FZ). The rest of the phase diagram is occupied by the chiral paramagnetic state (PM). The instability discussed in the text will lead to a modification of this scenario for all densities.

ing sliver of the plane also at high densities, being located close to the line

$$\bar{\alpha} = \frac{1}{r_s} + \frac{\pi - 1 - 2\mathcal{K}}{2\pi}, \tag{1.4}$$

where $\mathcal{K} \simeq 0.916$ is the Catalan constant. As it turns out all these states can be elegantly classified by means of one parameter, the generalized chirality $\chi$ which is defined in terms of the (dimensionless) radii $\kappa_{0\pm}$ of the circles delimiting the occupied regions in momentum space. When both chiral bands are occupied $\kappa_{0+(-)}$ corresponds to the Fermi radius of the larger (smaller) circle, while when only the lower chiral band is occupied $\kappa_{0+(-)}$ represents the outer (inner) radius of the occupied annulus. In the first case $\chi$ coincides with the standard chirality $\chi_0 = \frac{\kappa_{0+}^2 - \kappa_{0-}^2}{\kappa_{0+}^2 + \kappa_{0-}^2}$ while in the second it is larger than one, i.e.

$$\chi = \begin{cases} \chi_0 \,, & \text{for } 0 < \chi_0 < 1 \\ \dfrac{\kappa_{0+}^2 + \kappa_{0-}^2}{\kappa_{0+}^2 - \kappa_{0-}^2} \,, & \text{for } \chi_0 = 1 \,, \end{cases} \tag{1.5}$$

so that (in units of the Fermi wave vector $k_F = \sqrt{2\pi n}$) $\kappa_{0\pm} = \sqrt{|1 \pm \chi|}$. The situation can be readily visualized by inspecting Figures 2 and 4. Notice that both PM and FZ states have a renormalized momentum space occupation.

Consider now the situation at high densities. In this case as $\bar{\alpha}$ exceeds the value of Equation (1.4) the system appears to settle into a PM state in which only the lower chiral band is occupied and $\chi > 1$. Here, at first sight, it may appear safe to entertain the notion that by increasing the spin-orbit coupling at constant density (and therefore the strength of the non interacting part of the Hamiltonian) one would fall into the familiar paradigm in which the interacting part of the Hamiltonian becomes eventually irrelevant and therefore amenable to perturbative treatment. This is however not the case since, as we will presently show, for sufficiently large $\bar{\alpha}$ the effects of the Coulomb interaction are not perturbative. To demonstrate this effect we will construct a broken symmetry trial state and will show that its energy can be made lower than the corresponding interacting PM. In particular we will consider a state in which the momentum space occupation is repopulated in such a way as to break the circular symmetry in the $k_x, k_y$ space as depicted in Figure 4. To be specific we will construct a Slater determinant with occupation determined by the following Ansatz for the momentum occupation geometry (see Figure 4) [23]

$$\kappa_\pm(\phi) = \kappa_{0\pm} \pm \eta \cos\phi \,, \tag{1.6}$$

with the azimuthal direction of the spin quantization axis $\hat{s}_{\mathbf{k}} = \hat{\phi}_{\mathbf{k}}$ kept unchanged [24]. Since the distortion is infinitesimal, to lowest order the total energy change will depend on the value of the generalized chirality only. The energy change associated with Equation 1.6 can be calculated in the limit of large $\chi$. The kinetic plus spin-orbit energy change (in Rydbergs) is given by:

$$\delta\mathcal{E}_0 \simeq \left[\frac{2}{r_s^2} - \frac{\bar{\alpha}}{\sqrt{2}r_s}\left(\sqrt{\chi+1} - \sqrt{\chi-1}\right)\right]\eta^2 \,, \tag{1.7}$$

while, in the same units, for the exchange energy we find

$$\delta\mathcal{E}_x \simeq -\frac{4\chi \ln\chi}{\pi r_s \sqrt{2\chi}}\eta^2 \,. \tag{1.8}$$

Now, since for large $\bar{\alpha}$ we have $\chi \simeq \frac{(r_s\bar{\alpha})^2}{2}$ to leading order these competing energy contributions simplify to

$$\delta\mathcal{E}_0 \simeq \frac{1}{r_s^2}\eta^2 , \qquad \delta\mathcal{E}_x \simeq -\frac{4\bar{\alpha}\ln(\bar{\alpha}r_s)}{\pi}\eta^2 . \tag{1.9}$$

Thus for sufficiently large $\bar{\alpha}$ the differential instability is established.

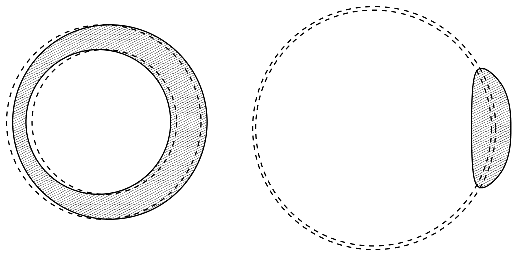

**Figure 4.** Left: Schematic of the symmetric unperturbed momentum space occupation for the paramagnetic state (area within the dashed annulus) and the asymmetric occupation corresponding to the in-plane ferromagnetic trial state (shaded area). Right: Case of large distortion in the limit of large $\bar{\alpha}$ value.

Clearly by thickening one side of the annulus the repopulation of Equation (1.6) leads to both an in plane momentum along the $x$-axis and a polarization along the $y$-axis. In the same regime of large values of $\chi$ and $\bar{\alpha}$ we find $P_x/N = \langle\sum_i \hat{p}_{x,i}\rangle/N \simeq \hbar k_F \chi\eta$ and $S_y/N = \langle\sum_i \hat{\sigma}_{y,i}\rangle/N \simeq \sqrt{\chi}\eta$. On the other hand the two quantities are balanced in such a way as to lead to a vanishing net velocity. In particular $V_x = P_x/mN - \alpha S_y/N = 0$. This clearly minimizes the energy.

Although the resolution of the instability is to be explored we have identified, by a consistency argument, the type of state that eventually takes over in the limit of very large $\bar{\alpha}$ or lower densities. This state corresponds to a fully polarized droplet in momentum space, as illustrated in the second panel of Figure 4. If one assumes that indeed the occupied region is centered about the wave vector $\mathbf{K} = \frac{m\alpha}{\hbar}\hat{x}$ (equal in magnitude to the radius of the occupied annulus, see Figure 4) then in the $r_s\bar{\alpha} \to \infty$ limit, using the fact that the spin quantization axes are asymptotically along $\hat{y}$, the functional (1.3) simplifies to

$$\mathcal{E} \simeq -\bar{\alpha}^2 + \frac{1}{\pi r_s^2}\int_{\mathcal{D}} k_x^2\, \mathrm{d}\boldsymbol{k} - \frac{\sqrt{2}}{(2\pi)^2 r_s}\int_{\mathcal{D}} \frac{\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}'}{|\boldsymbol{k}-\boldsymbol{k}'|} , \tag{1.10}$$

where we have used Rydberg units for the energy and the wave vectors are in units of $k_F$. Here the integrals are performed over the occupied region $\mathcal{D}$ of extension $2\pi$ (which we have folded from $\mathbf{K}$ back to the origin). Equation (1.10) describes confined classical charges interacting via an hard core potential that forces them to occupy the domain $\mathcal{D}$ and an attractive Coulomb potential in the presence of an additional external parabolic potential along the $x$ direction. The ensuing occupation consists of an oblate region, elongated in the $y$ direction. In the limit of large $\bar{\alpha}$ the actual shape becomes independent of this variable and is solely determined by the density parameter $r_s$. Since Equation (1.10) is valid when the linear size (approximately $k_F$) of the occupied region is small with respect to the radius $m\alpha/\hbar$, it can also be applied in the large $r_s$ limit at constant $\bar{\alpha}$. In this case the $1/r_s^2$ contribution can be neglected so that the consistent HF ground state corresponds to a fully polarized circular droplet of radius $\sqrt{2}k_F$ centered in $\mathbf{K}$. Also in this case the velocity vanishes. The energy of this state can be calculated exactly and it is given by:

$$\mathcal{E}^{(\mathrm{trial})} = -\bar{\alpha}^2 + \frac{2}{r_s^2} - \frac{16}{3\pi r_s} . \tag{1.11}$$

This result can be compared with the energy of the corresponding PM state which is given by

$$\mathcal{E}^{(\mathrm{PM})} \geq -\bar{\alpha}^2 - \frac{1.203}{r_s} , \tag{1.12}$$

where we used $-\bar{\alpha}^2$ as a lower bound for the kinetic and Rashba contributions, and the minimum unpolarized exchange energy [19], which occurs when the generalized chirality is $\chi \simeq 0.9147$. Clearly $\mathcal{E}^{(PM)} \geq \mathcal{E}^{(\mathrm{trial})}$ for $r_s \geq 4.044$, thus establishing the instability of the PM phase also in the low density limit.

It is important to realize that the physical underpinning of this symmetry breaking phenomenon can be attributed to the fact that as $\bar{\alpha}$ is increased, the occupied region in momentum space becomes an annulus of radius $\frac{m\alpha}{\hbar}$. Since the electron number is constant the annulus keeps getting thinner and, what is important, the bandwidth, approximately given (in Rydberg units) by $\frac{1}{r_s^4\bar{\alpha}^2}$ vanishes. This situation is depicted in the right panel of Figure 2. It is quite clear that this phenomenon is quite robust so that, while the description of the corresponding phase transition obtained via mean field theory should be considered as a rough approximation, correlation effects can only enhance the instability.

Although for any density the instability will occur for sufficiently large spin-orbit coupling, on the other hand if $\bar{\alpha}$ is kept constant, the Fermi liquid picture is recovered in the limit of high densities. We conclude by commenting that having established an in-plane ferromagnetic instability does not establish the HF phase diagram of the system. This can only be determined through a thoughtful numerical analysis.

## References

[1] G. F. GIULIANI and S. CHESI, "Highlights in the Quan- tum Theory of Condensed Matter", F. Beltram (editor), Edizioni della Normale, Pisa, 2005, p. 269.

[2] R. E. PEIERLS, "Quantum Theory of Solids", Oxford University Press, Oxford, 1955.

[3] B. BRAUNECKER, P. SIMON, and D. LOSS, Phys. Rev. B **80** (2009), 165119.

[4] B. BRAUNECKER, G. I. JAPARIDZE, J. KLINOVAJA and D. LOSS, Phys. Rev. B **82** (2010), 045127.

[5] C. P. SCHELLER, *et al.*, Phys. Rev. Lett. **112** (2014), 066801.

[6] CHESI, Ph.D. thesis, Purdue University, 2007.

[7] J. JUNG, M. POLINI and A. H. MACDONALD, arXiv:1111.1765 (2011).

[8] E. BERG, M. S. RUDNER and S. A. KIVELSON, Phys. Rev. B **85** (2012), 035116.

[9] T. A. SEDRAKYAN, A. KAMENEV and L. I. GLAZMAN, Phys. Rev. A **86** (2012), 063639.

[10] F. BLOCH, Z. Physik **57** (1929), 545.

[11] J. B. MILLER, D. M. ZUMBÜHL, C. M. MARCUS, Y. B. LYANDA-GELLER, D. GOLDHABER-GORDON, K. CAMPMAN and A. C. GOSSARD, Phys. Rev. Lett. **90** (2003), 076807.

[12] L. P. ROKHINSON, V. LARKINA, Y. B. LYANDA-GELLER, L. N. PFEIFFER and K. W. WEST, Phys. Rev. Lett. **93** (2004), 146601.

[13] L. MEIER, G. SALIS, I. SHORUBALKO, E. GINI, S. SCHÖN and K. ENSSLIN, Nat. Phys. **3** (2007), 650.

[14] G. F. GIULIANI and G. VIGNALE, "Quantum Theory of the Electron Liquid", Cambridge University Press, Cambridge, 2005.

[15] Some early work on the quasiparticle properties of this system can be found, in G.-H. CHEN and M. E. RAIKH, Phys. Rev. B **60** (1999), 4826; for a more recent analysis see D. S. SARAGA and D. LOSS, Phys. Rev. B **72** (2005), 195319.

[16] S. CHESI and G. F. GIULIANI, to be published.

[17] R. WINKLER, E. TUTUC, S. J. PAPADAKIS, S. MELINTE M. SHAYEGAN, D. WASSERMAN and S. A. LYON, Phys. Rev. B **72** (2005), 195321.

[18] S. CHESI and G. F. GIULIANI, Phys. Rev. B **75** (2007), 155305; in this paper one will also find the correct form taken by the spin-orbit

interaction in a number of experimentally relevant modern quantum well devices.
[19] S. CHESI, G. SIMION and G. F. GIULIANI, cond-mat/0702060 (2007).
[20] G. SIMION and G. F. GIULIANI, to be published.
[21] Y. A. BYCHKOV and E. I. RASHBA, JETP Lett. **39** (1984), 78; J. Phys. C **17** (1984), 6039.
[22] An equivalent Hamiltonian is obtained if the spin-orbit coupling is of the Dresselhaus type. See G. DRESSELHAUS, Phys. Rev. **100** (1955), 580.
[23] Notice that had we assumed different amplitudes for the distortions of $\kappa_\pm$ the trial state would have a finite current. It is easy to show that, not unexpectedly, the kinetic and spin-orbit energy cost is indeed minimized for identical magnitudes, *i.e.* for zero current.
[24] Relaxing this assumption does not improve the quality of the trial state in the regime studied. It is clear that for intermediate values of the parameters it will prove necessary to allow the spin orientations to relax.