Momentum Transfer and Effect of Motion of Atomic Mass Centre on an Atom-Cavity System

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Abstract For a special A-type atom with dark state, the problem of momentum transfer is discussed in detail. Effect of moving mass centre on atomic dynamics for various initial conditions is also studied. The numerical results show that the Doppler effect exercised by the motion of atomic mass centre can lead to the modified phenomenon of oscillation collapse and revival in the transition probability and the atomic population inversion.

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The Jaynes–Cummings model (JCM)\textsuperscript{[1]} for dynamics of the two-level atom interacting with a single-mode radiation field is one of the exactly solvable quantum mechanical models in quantum optics. The original JCM\textsuperscript{[1]} only characterizes the direct interaction between the atomic levels and the cavity field by assuming that the atom only “sees” a homogeneous electromagnetic field in a cavity at its own scale.

Recently, new developments in the techniques associated with quantum optics have made it possible to produce a cavity with an extremely small size comparable to the wavelength of atomic emissions\textsuperscript{[2,3]} in various cavities with so small size, many distinctive phenomena of atomic motion appear to be attractive for both the interests in fundamental theory and the advances in new high techniques. These phenomena mainly concern the suppression and enhancement of atomic spontaneous radiation by cavity,\textsuperscript{[3]} the deflection of atoms connected with the potential realization of atomic interferometers, the cooling of atoms by an adiabatically-decaying cavity\textsuperscript{[4–6]} and so on. It is not difficult to see that the effect of the spatial motion of atomic centre on the “internal energy level” rules the above-mentioned features of atom-field interaction system through an inhomogeneous cavity field at the scale of atom.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{energy_level.png}
\caption{Energy-level structure for the system considered here. The upper level |e\rangle is connected through one-photon transition to the two lower levels |g_1\rangle and |g_2\rangle.}
\end{figure}

To study the effect of spatial motion of atomic mass centre, a generalized JCM is present for a two-level atom interacting with a locally-inhomogeneous single-mode cavity field.\textsuperscript{[7]} In fact,

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the effect of spatial motion of atomic mass centre is associated with momentum transferred as
the typical phenomena in the so-called atomic optics. The present paper will emphasize this
effect for a type of three-level atoms, the Λ-type atom as shown in Fig. 1. Here, the energy
eigenvalue of two ground states \( |g_1\rangle \) \((i = 1, 2)\) and excited state \( |e\rangle \) are \( \hbar \omega_i \) \((i = g_1, g_2, e)\)
respectively and the transition between \( |g_2\rangle \) and \( |g_1\rangle \) is inhibited. Such a type of three-
level atom has been used to analyze the coherent population trapping for the case without
considering the motion of atomic mass centre.[8]

In this paper, the problem of momentum transfer is discussed in detail for a special Λ-type
atom with dark state. The back action of atomic mass centre on both the cavity field and
internal levels is also studied. By solving the evolution equation of the model numerically, the
spatial motion will cause some new dynamical features, the modifications of phenomenon of
oscillation collapse and revival in atomic transition probability and population inversion.

We consider a three-level atom with two ground states \( |g_1\rangle \) and \( |g_2\rangle \) coupled to an excited
state \( |e\rangle \) via, respectively, a classical laser field \( \Omega_1(t) \) and a cavity-mode field of frequency \( \omega \).
For the moment we neglect atomic spontaneous emission and cavity damping. The Hamilton-
ian for this system can thus be written as
\[
H = \hbar \omega t \hat{a} + \frac{\hbar^2}{2m} + \hbar \omega_1 |g_1\rangle \langle g_1| + \hbar \omega_2 |g_2\rangle \langle g_2| + \hbar \Omega_1(t) \left[ e^{-i \hbar \frac{1}{2} x \hat{a}^\dagger \hat{a} |g_1\rangle \langle g_1| + e^{i \hbar \frac{1}{2} x \hat{a}^\dagger \hat{a} |g_1\rangle \langle g_1| \right] + \hbar \Omega_2(t) \left[ \hat{a}^\dagger e^{-i \hbar \frac{1}{2} x \hat{a}^\dagger \hat{a} |g_2\rangle \langle g_2| + e^{-i \hbar \frac{1}{2} x \hat{a}^\dagger \hat{a} |g_2\rangle \langle g_2| \right]
\]
(1)
where \( \hat{a} \) and \( \hat{a}^\dagger \) denote respectively the annihilation and creation operators of bosonic cavity
mode with frequency \( \omega \); \( \hat{p} \) is the momentum operator conjugating to the position \( x \) along the
propagating axis of the cavity field; \( k_2 \) is the wave number for running wave of cavity and \( k_1 \)
is that for the classical laser field; \( \Omega_i \) \((i = 1, 2)\) is the coupling constant.

By invoking a unitary transformation
\[
\tilde{W}(x) = |e\rangle \langle e| + e^{-i \hbar \frac{1}{2} x \hat{a}^\dagger \hat{a} |g_1\rangle \langle g_1| + e^{-i \hbar \frac{1}{2} x \hat{a}^\dagger \hat{a} |g_2\rangle \langle g_2|)
\]
(2)
one obtains an equivalent Hamiltonian \( \tilde{H}_e = \tilde{W}^\dagger(x) \tilde{H} \tilde{W}(x) \)
\[
\tilde{H}_e = \hbar \omega_1 |g_1\rangle \langle g_1| + \hbar \omega_2 |g_2\rangle \langle g_2| + \hbar \Omega_1(t) \left[ |e\rangle \langle g_1| + |g_1\rangle \langle e| \right] + \hbar \Omega_2(t) \left[ \hat{a}^\dagger |g_2\rangle \langle e| + |g_1\rangle \langle g_2| \right]
\]
(3)
where
\[
\omega'_1 = w_1 + \frac{p k_1}{m} + \frac{\hbar^2 k_1^2}{2m}, \quad \omega'_2 = w_2 + \frac{p k_2}{m} + \frac{\hbar^2 k_2^2}{2m}
\]
The effective frequencies \( \omega'_i \) \((i = 1, 2)\) were modified by the Doppler shift \( k_i \hat{p}/m \) \((i = 1, 2)\) and
photon recoil \( \hbar^2 k_i^2/(2m) \) \((i = 1, 2)\).

For a special case that \( \omega'_1 = \omega'_2 = \omega = \omega_p \), we have a matrix representation of the Hamiltonian \( \tilde{H}_e \) in the invariant subspace spanned by \( |g_1, n\rangle \), \( |g_2, n + 1\rangle \) and \( |e, n\rangle \)
\[
\tilde{H}_e = n \hbar \omega + \hbar \omega_p + \frac{p^2}{2m} + \begin{bmatrix}
\hbar \omega_e & \hbar \Omega_1 & \hbar \Omega'_1 \\
\hbar \Omega_1 & 0 & 0 \\
\hbar \Omega'_1 & 0 & 0
\end{bmatrix}
\]
(4)
where
\[
\Omega'_2 = \Omega_2 \sqrt{n + 1}
\]
(5)
If we introduce
\[
\Omega = \sqrt{\left(\omega_e - \omega_p\right)^2 + 4 \Omega_1^2 + 4 \Omega_2^2}, \quad \Omega_1 = \frac{1}{2} \Omega \sin \theta \cos \phi, \quad \Omega_2 = \frac{1}{2} \Omega \sin \theta \sin \phi,
\]
(6)
then
\[
\tilde{H}_e = n \hbar \omega + \frac{p^2}{2m} + \hbar \omega_p + \hbar \Omega \begin{bmatrix}
\cos \theta & \left(\sin \theta \cos \phi\right)/2 & \left(\sin \theta \sin \phi\right)/2 \\
\left(\sin \theta \cos \phi\right)/2 & 0 & 0 \\
1/2 & 0 & 0
\end{bmatrix}
\]
(7)
where \( \tan \phi = \Omega_1/\Omega_2 \), \( \tan \theta = 2\sqrt{\Omega_1^2 + \Omega_2^2}/(n\omega + \omega_e - \omega_p) \). Then, the eigenstates and eigenvalues can be expressed in concise forms

\[
\begin{align*}
|\psi_{0n}\rangle &= \sin \phi |g_1, n\rangle - \cos \phi |g_2, n + 1\rangle, \\
|\psi_{+n}\rangle &= \cos \frac{\theta}{2} |e, n\rangle + \sin \frac{\theta}{2} (\cos \phi |g_1, n\rangle + \sin \phi |g_2, n + 1\rangle), \\
|\psi_{-n}\rangle &= \sin \frac{\theta}{2} |e, n\rangle - \cos \frac{\theta}{2} (\cos \phi |g_1, n\rangle + \sin \phi |g_2, n + 1\rangle)
\end{align*}
\]  
(8)

and

\[
E_{0n} = n\hbar \omega + \frac{p^2}{2m} + \hbar \omega_p,
\]

\[
E_{\pm n} = \hbar \Omega \left( \frac{\cos \frac{\theta \pm 1}{2}}{2} \right) + n\hbar \omega + \hbar \omega_p + \frac{p^2}{2m}.
\]  
(9)

The above solution shows that the \( \Lambda \)-atom with special case \( \omega_1' = \omega_2' + \omega \) is just “dark atom” with a dark state \(|\psi_{0n}\rangle\) decoupling with the excited state \(|e\rangle\). This kind of \( \Lambda \)-atom can be used as the transfer of both population and momentum in a multi-level system by adiabatic following of a slowly evolving light.\(^9\) In fact, if we switch off the interaction by letting \( \Omega_1 = 0 \) between \(|e\rangle\) and \(|g_1\rangle\) adiabatically for the \( \Lambda \)-atom prepared initially in \(|\phi_{0n}\rangle = |g_1, n\rangle \) (\( \Omega_1 \gg \Omega_2 \)), then the system will be evolved into another state \(|g_2, n + 1\rangle\). This is just the mechanism for transfer of population. To understand the process of transfer of momentum, the momentum state \(|p\rangle\) of atomic mass centre must be included into the energy eigenstate. Transforming back to original representation through \( \hat{W}(x)^{-1} \), we have

\[
|\psi_{0n}(p)\rangle = \hat{W}(x)^{-1}(|\psi_{0n}\rangle \otimes |p\rangle) = \sin \phi |g_1, n\rangle \otimes (|p\rangle - \hbar \vec{k}_1) - \cos \phi |g_2, n + 1\rangle \otimes |p\rangle - \hbar \vec{k}_2). \]  
(10)

This means that the above-mentioned adiabatic shuttling-down will result in momentum transfer by \( \hbar \vec{k}_2 - \hbar \vec{k}_1 \). The state \(|\psi_{0n}\rangle\) exhibits the following asymmetric behavior as a function of time

\[
|\psi_{0n}\rangle \longrightarrow \begin{cases} 
|g_1, n\rangle & \text{for } \Omega_1/\Omega_2 \rightarrow 0, \\
|g_2, n + 1\rangle & \text{for } \Omega_2/\Omega_1 \rightarrow 0.
\end{cases}
\]  
(11)

When the time evolution to the system is adiabatic, equation (11) demonstrates that the passage of an atom through the cavity and laser field produces a single-photon “shift” of the cavity mode photon distribution which is accomplished with a momentum transfer. For example, if, as the atom enters the interaction region, the state of the atom-cavity system is given by \(|g_1, n\rangle \) (\( \Omega_1 \gg \Omega_2 \)), then, \( \Omega_1 \) is time delayed with respect to \( \Omega_2 \), the final state of the system as the atom exits the interaction region will be \(|g_2, n + 1\rangle\). Obviously, the transit of \( N \) such atoms shifts the distribution by exactly \( N \) photons. For the case of an initial vacuum state of the cavity mode, \( \rho_F = |0\rangle\langle 0| \), this corresponds to the generation of an \( N \)-photon Fock state \( \rho_F = |N\rangle\langle N| \), provided, of course, that cavity losses are negligible during the interval between the first and last atoms.

In fact, the condition \( \omega_2' = \omega_1' \), under which the dark state forms, depends on the momentum \( p \) of the atomic mass centre due to the Doppler effect and the photon recoil. If the above-mentioned condition is not valid, the solution cannot be expressed in a concise form, and we need to use the numerical solution of the effect Hamiltonian \( H_e \). In an invariant subspace spanned by \(|e\rangle \otimes |n\rangle\), \(|g_1\rangle \otimes |n\rangle\) and \(|g_2\rangle \otimes |n + 1\rangle\), the matrix representation of the Hamiltonian \( H_e \) is

\[
\frac{p^2}{2m} + n\hbar \omega + \begin{bmatrix}
\hbar \omega_e & \hbar \Omega_1 & \hbar \Omega_2 \sqrt{n + 1} \\
\hbar \Omega_1 & \hbar \omega_1' & 0 \\
\hbar \Omega_2 \sqrt{n + 1} & 0 & \hbar \omega_2' + \hbar \omega
\end{bmatrix},
\]  
(12)
where we assume that the momentum of mass centre has definite momentum $p$. Diagonalizing this matrix numerically, the eigenstates of the Hamiltonian $H_e$ are given by
\[ |\chi_1\rangle = a_1|e\rangle \otimes |n\rangle + b_1|g_1\rangle \otimes |n\rangle + c_1|g_2\rangle \otimes |n + 1\rangle, \]
\[ |\chi_2\rangle = a_2|e\rangle \otimes |n\rangle + b_2|g_1\rangle \otimes |n\rangle + c_2|g_2\rangle \otimes |n + 1\rangle, \]
\[ |\chi_3\rangle = a_3|e\rangle \otimes |n\rangle + b_3|g_1\rangle \otimes |n\rangle + c_3|g_2\rangle \otimes |n + 1\rangle, \]
(13)
and $E_1$, $E_2$ and $E_3$ are the corresponding eigenvalues respectively. Here $a_i$, $b_i$, $c_i$, $E_i$ ($i = 1, 2, 3$) depend upon $\omega$, $\omega_1$, $\omega_2$, $\omega_e$, $\Omega_1$, $\Omega_2$, $n$ and $p$. In the subsequent numerical calculation we assume $\omega = 0.2$, $\omega_1 = 0.01$, $\omega_2 = 0$, $\Omega_1 = \Omega_2 = 1.0$, $k_1 = k_2 = m$, $h = 1$, $\omega_e = 0.2$.

Let us use the above solutions to study the dynamic effects of spatial motion of mass centre, the population inversion and the transition probability. In principle, to describe dynamics of a generalized JCM, one must select one of various initial conditions corresponding to different statistical properties, as different initial conditions can result in remarkable differences in the dynamical characters of cavity-atom system. Let us first consider the case with the initial state of field having exactly $N$ photons and the momentum of mass centre for the atom in ground state $|g_2\rangle$ obeying the Gaussian distribution
\[ |c(p)|^2 = \sqrt{\frac{2a^2}{\pi}} e^{-2a^2(p-p_0)^2/\hbar^2} \]
(14)
With this initial condition, the wavefunction follows from Eq. (13)
\[ |\phi(t)\rangle = \frac{1}{c_1 + c_2x + c_3y} \int_{-\infty}^{\infty} c(p)dp\left\{ [\eta_1 a_1 + xa_2\eta_2 + ya_3\eta_3] |g_1\rangle \otimes |N\rangle \otimes |p - \hbar k_1\rangle + \right. \]
\[ \left. [\eta_1 b_1 + xb_2\eta_2 + yb_3\eta_3] |e\rangle \otimes |N + 1\rangle \otimes |p\rangle + \right. \]
\[ \left. [\eta_1 c_1 + xc_2\eta_2 + yc_3\eta_3] |g_2\rangle \otimes |N\rangle \otimes |p - \hbar k_2\rangle \right\}, \]
(15)
where
\[ x = \frac{b_1a_3 - b_3a_1}{b_2a_3 - b_3a_2}, \quad y = \frac{b_2a_1 - a_2b_1}{b_2a_3 - b_3a_2}, \]
\[ \eta_1 = \exp\left[-\frac{i}{\hbar} E_1 \left(p - \frac{\hbar k_1}{2}\right)t\right], \quad \eta_2 = \exp\left[-\frac{i}{\hbar} E_2 \left(p - \frac{\hbar k_2}{2}\right)t\right], \quad \eta_3 = \exp\left[-\frac{i}{\hbar} E_3 \left(p - \frac{\hbar k_2}{2}\right)t\right]. \]
Thus the probability of transition from $|g_2, N + 1\rangle$ to $|e, N\rangle$ is obtained by tracing over the spatial variable as
\[ P = |\langle e, N |\phi(t)\rangle|^2 = \int_{-\infty}^{\infty} dp |c(p)|^2 \left(\frac{1}{c_1 + c_2x + c_3y}\right)^2 \left[(a_1 \cos E_1 t + a_2 \cos E_2 t + a_3 \cos E_3 t)^2 + (a_1 \sin E_1 t + a_2 \sin E_2 t + a_3 \sin E_3 t)^2\right]. \]
(16)
We may remark that if the initial momentum has the value $p$, then the ground-state component of atomic evolution state has a momentum shift $\hbar k_2 - \hbar k_1$. This shift will lead to the split of atomic beam and/or pulse.

The result of numerical calculation of Eq. (8) is shown in Fig. 2 where we assume $N = 5$, $p_0 = 2$. Figure 2 shows the behavior or transition from $|g_2, N + 1\rangle$ to $|e, N\rangle$. As the breadth becomes wider (Figs 2a–2c). Note that, very large $a$ means the zero breadth of the wave packet in momentum space and the atom has a definite momentum.), the different oscillation components corresponding to frequencies determined by different $p$ form a superposition to realize an oscillation collapse. In comparison with the original JCM model, the revival is not evident due to Doppler effect. The heights of revival signals decrease and the long-time behavior of the transition becomes increasingly random (Fig. 2d).
Fig. 2. Transition probabilities from $|g_2, N + 1\rangle$ to $|e, N\rangle$ with $g = 1$, $p_0 = 2$ and the initial momentum of atomic mass centre having a Gaussian distribution. Figures 2a–2c correspond to different $a$, $a = 40, 5, 1$, respectively. Figure 2d is long-time behavior of the transition probability with $a = 10$.

Furthermore, we study the effect of moving mass centre on the population inversion. To clarify the discussion we will modify the Hamiltonian (1) to the intensity-dependent case,$^{[10]}$ i.e., let $a \rightarrow a(a^\dagger a)^{1/2}$, $a^\dagger \rightarrow (a^\dagger a)^{1/2}a^\dagger$, as we stated above, Doppler and recoil shifts of the frequency have influence on the evolution of expectation value of atomic internal energy, if the cavity is initially in a coherent state $|\alpha\rangle$, we have

$$E(t) = \langle \phi(t)|\hbar \omega_1|e_1\rangle\langle e_1| + \hbar \omega_2|e_2\rangle\langle e_2|\phi(t)\rangle = \hbar \omega_0 \exp(-|\alpha|^2) \int_{-\infty}^{\infty} dp |c(p)|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \times$$
\[
\left\{ \frac{1}{c_1 + c_2 x + c_3 x} \right\}^2 [\hbar \omega_1 (a_1 \cos E_1 t + a_2 \cos E_2 t + a_3 \cos E_3 t)^2 + \hbar \omega_2 (b_1 \cos E_1 t + b_2 \cos E_2 t + b_3 \cos E_3 t)^2 + \hbar \omega_2 (b_1 \sin E_1 t + b_2 \sin E_2 t + b_3 \sin E_3 t)^2] \times
\left\{ \frac{1}{c_1 + c_2 x + c_3 x} \right\}^2
\left\{ \frac{1}{c_1 + c_2 x + c_3 x} \right\}^2.
\]

(17)

Figure 3 shows the time evolution of the population inversion at the non-resonance cases with different \(a\). It is shown that the phenomena of revival and oscillation collapse appear in the presence of the Doppler effect, but the heights of revival signals decrease, when the wave packet becomes wider (Fig. 3b). Hence, in this case, the phenomena of collapse and revival are not evident in comparison with that in the original JC model, the phenomena of collapse and revival disappear while \(a\) decreases (Fig. 3b). This is similar to the case of the original JCM with coherent state cavity field when the mean photon number is increased.[11]

![Graph](image)

**Fig. 3.** The population inversion for a cavity initially in a coherent state and the initial momentum of mass centre having a Gaussian distribution. The mean photon number is 3, and the other parameters are the same as Fig. 1. Figures 3a and 3b correspond to different \(a\), \(a = 3.0\), \(a = 0.3\) respectively.

**References**