Two mode photon bunching effect as witness of quantum criticality in circuit QED

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We suggest a scheme to probe critical phenomena at a quantum phase transition (QPT) using the quantum correlation of two photonic modes simultaneously coupled to a critical system. As an experimentally accessible physical implementation, a circuit QED system is formed by a capacitively coupled Josephson junction qubit array interacting with one superconducting transmission line resonator (TLR). It realizes an Ising chain in the transverse field (ICTF) which interacts with the two magnetic modes propagating in the TLR. We demonstrate that in the vicinity of criticality the originally independent fields tend to display photon bunching effects due to their interaction with the ICTF. Thus, the occurrence of the QPT is reflected by the quantum characteristics of the photonic fields.

quantum phase transition, photon bunching, circuit QED

Inspired by the fast developments of quantum information⁴⁵, quantum phase transition (QPT)⁶ has renewed much attention in different fields of physics ranging from condensed matter physics to quantum optics⁴⁵. Its close relation with entanglement was well explored in spin models⁶⁷. It was found that⁸ at the quantum critical point the dynamic evolution of a quantum critical system is so extremely sensitive that it can enhance the quantum decoherence of an external system coupled to it. This ultra-sensitivity is characterized by the Loschmidt echo, which is a well-known concept in quantum chaos⁹. In this sense, the quantum-classical transition from a pure state to a mixed one is induced by the quantum criticality of this surrounding system. This discovery motivated a new scheme to probe the QPT by exploring the quantum coherence in the external system and its losses¹⁴.

Moreover, this probing mechanism for quantum criticality was illustrated by a circuit QED architecture¹⁰⁻¹², which was formed by a superconducting Josephson junction qubit array interacting with a one-dimensional superconducting transmission line resonator (TLR)¹³. The superconducting qubit array was modeled as an Ising chain in the transverse field (ICTF). This investigation showed that the QPT phenomenon in the superconducting qubit array was evidently revealed by the correlation spectrum of TLR output. Though this mechanism for the circuit QED system has not been experimentally tested, an NMR simulation experiment¹⁴ has been carried out to demonstrate the QPT-like phenomenon (energy level crossing) as predicted in ref. [8] by exploring the increased sensibility of the quantum system to perturbation when it is close to a critical point.

For the above circuit QED architecture to demon-
strate the probing of the QPT, we notice that with two
modes simultaneously coupled to a charge qubit, their
squeezing effect was investigated theoretically. Here,
we consider the full application of quantum optics
approach in the detection of the QPT by considering the
higher order quantum coherence. To this end, we con-
sider that a Josephson junction qubit array modeled as
the ICTF simultaneously couples to two modes propa-
gating in the TLR. Since all quasi-spins homogeneously
interact with the fields, we can obtain the first (second)
order correlation function of the two fields. According
to our calculation, the second order quantum coherence is
given in terms of the decoherence factor of the ICTF. As
proven in Appendix A, the norm of the decoherence
factor decreases exceptionally when the ICTF is at the
critical point. Therefore, the photon bunching effect oc-
curs since the second order quantum coherence of the
steady state is smaller compared with its initial value.
And these results show genetic characteristics of the
quantum spin chain in the vicinity of the critical point.

The paper is structured as follows. The next section
describes the ICTF formed by a capacitively coupled
Josephson junction qubit array is coupled to two inde-
pendent fields propagating in the TLR. Then the detec-
tion scheme of the QPT and the correlation functions of
two mode fields are given in Sec 2. A brief summary is
given in Sec 3. Furthermore, in addition to the main
body of the paper, Appendix A presents the details about
the calculation of the decoherence factor. Since the
theoretical deduction is based on the rotating wave ap-
proximation (RWA), its validity has been proven in
Appendix B.

1 Circuit QED based setup for probing
quantum criticality

We consider a circuit QED system illustrated in Figure 1.
N Cooper pair boxes (CPBs) are capacitively coupled
one by one. Formed by a superconducting island con-
nected with two Josephson junctions, each CPB is a di-
rect current superconducting quantum interference de-
vice (dcSQUID). Since the magnetic flux \( \Phi_x \) threading
the dcSQUID is tunable, the effective Josephson tunnel-
ing energy can be varied. With proper bias voltage, the
CPB behaves as a qubit near the degeneracy point and
then Josephson junction qubit array becomes a spin
chain with \( N \) 1/2-spins. When the coupling capacitance
\( C_m \) between two CPBs is much smaller than the total one

\[
C_2 \text{ to each CPB, the high order terms in Hamiltonian can be}
\text{neglected and only the nearest neighbor interaction is}
\text{considered. Then the qubit array can be approximated as}
\text{an ICTF with ref. [14]}:
\]

\[
H_0 = B \sum_{j=1}^{N} \lambda \sigma_j^x + \sigma_j^z \sigma_{j+1}^z,
\]

where \( \sigma^x = |0\rangle \langle 1| - |1\rangle \langle 0| \) and \( \sigma^z = |0\rangle \langle 0| - |1\rangle \langle 1| \) with
\( |n\rangle \) being the state of \( n \) extra Cooper pairs on the super-
conducting island, \( \lambda = B_x / B \) and \( B = e^2 C_m / C_2^2 \),
\( B_x = E_j \cos(\pi \Phi_x / \Phi_0) \) is the Josephson energy of each
CPB with \( E_j \) the Josephson energy of single junction,
\( \Phi_0 = h / 2 e \) the flux quantum.

In a one dimensional TLR, the electric current and
voltage at the position \( x \) are given as

\[
I(x,t) = \sum_k \frac{\hbar \omega_k}{2 \pi} (a_k^+ + a_k) \sin \frac{k \pi x}{L},
\]

\[
V(x,t) = -\sum_k \frac{\hbar \omega_k}{2 \pi c} (a_k^+ - a_k) \cos \frac{k \pi x}{L},
\]

where \( a_k^+ \) is the creation operator with frequency \( \omega_k \),
\( L \) the length of TLR, \( l \) and \( c \) the induction and capaci-
tance of per unit length of TLR respectively, \( k \) positive
integer. Therefore, a CPB located at the antinode is only
coupled to the magnetic field since the electric field
vanishes. According to Ampere’s circuital law, when a
dc SQUID loop is placed at a distance \( r \) with respect to
the center of the TLR, the quantum magnetic flux that
threads it is

\[
\Phi_\ell(x) = \frac{\mu_0 S}{2 \pi r} \sum_k \frac{\hbar \omega_k}{L c} (a_k^+ + a_k) \sin \frac{k \pi x}{L},
\]

where \( \mu_0 \) is the vacuum magnetic permeability, \( S \) the

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\]

where \( \mu_0 \) is the vacuum magnetic permeability, \( S \) the
area of dc SQUID loop. The interaction between the CPBs and the magnetic field is written as

$$H_\phi = E_1 \sum_j \cos \frac{\pi \Phi_j(x_j)}{\Phi_0} \sigma_j^x.$$  \hspace{1cm} (5)

In our consideration, two independent modes with frequencies $\omega_0 = 3\omega_2$ are propagating in the TLR. All the CPBs are placed the antinodes of the both modes with the positions

$$x_j = \frac{j - \frac{1}{2}}{\omega_2} \pi v,$$  \hspace{1cm} (6)

where

$$\omega_2 = \frac{\pi vL}{M},$$  \hspace{1cm} (7)

$j$ and $M$ are positive integers, $v$ the velocity of the light. Since $\Phi_0 \ll \Phi_0$, under the RWA\cite{10}, the interaction Hamiltonian is approximated to the second order,

$$H_\phi = E_1 \sum_j \left[ \frac{1-\frac{1}{2}}{2} \left( \eta_1^3 (2a_i^+a_i + 1) + \eta_2^3 (2a_2^+a_2 + 1) \right) \right] \sigma_j^x,$$  \hspace{1cm} (8)

where the coupling constants between the two modes and individual spins are

$$\eta_i = \frac{\pi \mu_0 S}{2 \pi r \Phi_0} \sqrt{\frac{h \omega_0}{E_1}}.$$  \hspace{1cm} (9)

For realistic parameters, $C_x = 600 \text{ aF}$, $C_m = 30 \text{ aF}$, $L_0 = 1 \text{ cm}$, $S_0 = 10 \mu \text{m}^2$, $r = 1 \mu \text{m}$, $N = 500$, $E_1 = 13 \text{ GHz}$, we have $B = 1.6 \text{ GHz}$, $\omega_2 \approx 120 \text{ GHz}$, $\eta_1 = \sqrt{3}\eta_2$, and $\eta_2 \approx 0.1$.\cite{14}

Thus, the total Hamiltonian is written as

$$H = \omega_0 a_i^+a_i + \omega_2 a_2^+a_2 + E_1 \sum_j \left[ \frac{1-\frac{1}{2}}{2} \left( \eta_1^3 (2a_i^+a_i + 1) + \eta_2^3 (2a_2^+a_2 + 1) \right) \right] \sigma_j^x + \frac{\epsilon C_x}{C_m} \sum_j \sigma_j^x \sigma_j^z.$$  \hspace{1cm} (10)

The validity of the RWA is proven in Appendix B. Furthermore, since there is no energy exchange between the fields and the ICTF, the total Hamiltonian can be decomposed into invariant subspaces with respect to the Fock state of the fields,

$$H = \sum_{m,n} H^{(m,n)} |m\rangle \langle m|,$$  \hspace{1cm} (11)

where

$$H^{(m,n)} = B \sum_{j=1}^N \lambda_{m,n} \sigma_j^+ + \sigma_j^- \sigma_j^+,$$  \hspace{1cm} (12)

with $\lambda_{m,n} = E_1 [1 - \eta_1^3 (m + 1/2) + \eta_2^3 (n + 1/2)] / B$.

Generally speaking, the Hamiltonian of ICTF $H_0$ is transformed into a quadratic Fermion form with Jordan-Wigner transformation\cite{13}

$$c_j = \exp \left[ \sum_{i=k}^{j-1} \frac{\pi i}{2} \sigma_i^z \right] \sigma_j^x.$$  \hspace{1cm} (13)

Then, by introducing quasi-particle operator\cite{17}

$$\gamma_k = \sum_{j=1}^N \frac{e^{-\alpha j}}{\sqrt{j}} \left( c_j \frac{\theta_k}{2} - i \sin \frac{\theta_k}{2} c_j^+ \right)$$  \hspace{1cm} (14)

with

$$\theta_k(\lambda) = \tan^{-1} \frac{\sin k}{\lambda - \cos k},$$  \hspace{1cm} (15)

$H_0$ is diagonalized as

$$H_0 = \sum_k E_k \left( \gamma_k^+ \gamma_k - \frac{1}{2} \right)$$  \hspace{1cm} (16)

with single particle spectrum being

$$E_k(\lambda) = 2B\sqrt{1 + \lambda^2 - 2\lambda \cos k}.$$  \hspace{1cm} (17)

And the ground state $|G\rangle$ corresponds to no quasiparticle excitation at all.

## 2 Photon bunching effect

Followed by a series of advances, i.e., resonance fluorescence, the Hanbury-Brown-Twiss experiment\cite{18} opens philosophical debate about photons\cite{19} and sets itself as the milestone in the development of quantum optics. All these experimental phenomena are associated with the correlation functions of the field. Here, we consider it as the method to detect the QPT since the two fields propagating in the TLR interact with the quasi-spins respectively.

First of all, we define an operator

$$A = a_i + ia_2.$$  \hspace{1cm} (18)

The first order correlation function is written as $\langle A^+(t)A \rangle$. Here, the bracket $\langle \cdot \cdot \cdot \rangle$ denotes average over the initial state, with the Ising chain in the ground state $|G\rangle$ and the two fields being in arbitrary pure states $\sum_m c_m |m\rangle$ and $\sum_n d_n |n\rangle$ respectively. Therefore,

$$\langle A^+(t)A \rangle = \sum_{m,n} |c_m|^2 |d_n|^2 (m^{(m,n)}_{m-1,n} + n^{(m,n)}_{m,n-1})$$

$$+ \sum_{m,n} \rho^{(m,n)}_{m-1,n} d_n^* \langle m(n+1) | (m-1,n+1) \rangle$$

$$+ \sum_{m,n} \rho^{(m,n)}_{m,n-1} d_n^* \langle (m+1) | (m,n-1) \rangle.$$  \hspace{1cm} (19)
where
\[ r^{(m,n)}_{m,n}(t) = \langle G| \phi^{H(m,n)} e^{-iH_{c}(m,n) t/\gamma} |G \rangle \]
(20)
is the decoherence factor\(^{23}\) which measures the overlap of the ground state evolving under two different Hamiltonians. Details about its calculation is presented in Appendix A. In ref. \[12\], it was discovered that for the same amount of environment dissipation the first order correlation function of the single mode decreases more rapidly in the vicinity of the QPT than in the other region. Moreover, the second order correlation function is analytically written as
\[ \left\langle A^{+}A(t)A(t)A \right\rangle = \sum_{m,n} \left| c_{m} \right|^{2} \left| d_{n} \right|^{2} \left[ (m+n)(m+n-1) \right. \]
+ \left. m \sqrt{(m+1)n} \right] e^{-i(\omega_{m} \gamma t)} + n \sqrt{(n+1)m} \right]
(21)

Thus, for the fields initially in the state \((\ket{0} \pm \ket{1}) / \sqrt{2}\), it is straightforward to obtain
\[ \left\langle A^{+}A(t)A(t)A \right\rangle = \frac{1}{2} \left[ 1 + \text{Re}(\lambda_{0,1} e^{i(\omega_{0} - \omega_{1}) t}) \right], \]
(22)
where \(\text{Re}(x)\) means the real part of \(x\).

As proven in Appendix A, in the vicinity of the QPT, the square of the norm of \(\lambda_{0,1}^{(0,0)}(t)\) decreases more rapidly than exponential, i.e.,
\[ \left| \lambda_{0,1}^{(0,0)}(t) \right|^{2} \leq e^{-\gamma \tau}. \]
(23)

where \(\gamma = 4B^{2}(\delta_{0,0} - \delta_{0,1})^{2} E(k_{c}) / (2\lambda_{0,1} - 1)^{2}\), \(E(k_{c}) = 4\pi N_{c}(N_{c} + 1)(2N_{c} + 1) / 6N^{2}\) with \(N_{c}\) being the nearest integer to \(Nk_{c} / 2\pi\).

It can be seen that there is a vanishing numerator \(E(k_{c})\) as \(N \to \infty\). It is doubtful that the exponential decay of \(\left| \lambda_{0,1}^{(0,0)}(t) \right|^{2}\) can truly occur since the QPT takes place in the thermodynamical limit. However, of the ICTF gets larger, we can adjust the parameter \(\lambda_{0,1}\) closer to the critical point to make the denominator \((2\lambda_{0,1} - 1)^{2}\) small enough. In that case, \(\gamma\) stays as a constant and the \(\left| \lambda_{0,1}^{(0,0)}(t) \right|^{2}\) decreases exponentially with time. For a real system, \(N\) is finite for the demonstration of the QPT. To test the validity of the above analysis, we resort to numerical simulation. In Figure 2, we plot the evolution of \(\left| \lambda_{0,1}^{(0,0)}(t) \right|^{2}\) according to eq. (a9). It can be seen that despite some oscillations \(\left| \lambda_{0,1}^{(0,0)}(t) \right|^{2}\) decays exceptionally at the critical point.

According to ref. \[16\], the photon bunching and antibunching effects are associated with the second order degree of coherence
\[ g^{(2)}(t) = \left\langle A^{+}A(t)A(t)A \right\rangle / \left\langle A^{+}A \right\rangle \left\langle A \right\rangle, \]
(24)
which is the normalized second order correlation function of the fields. For the fields both in the state \((\ket{0} \pm \ket{1}) / \sqrt{2}\), the second order degree of coherence is simplified as
\[ g^{(2)}(t) = \frac{1 + \text{Re}(\lambda_{0,1} e^{i(\omega_{0} - \omega_{1}) t})}{2 - \text{Im}(\lambda_{0,1} e^{i(\omega_{0} - \omega_{1}) t})}, \]

with \(\text{Im}(x)\) being the image part of \(x\).

Since the norm of the decoherence factor decreases exponentially at the critical point, it is obvious that both the real and image parts of \(\lambda_{0,1}^{(0,0)}(t)e^{i(\omega_{0} - \omega_{1}) \gamma t}\) will vanish in that limit. As a consequence, we expect the second order degree of coherence to be less than unity in the steady state, i.e., \(g^{(2)}(t) = 1 / 2 < g^{(2)}(0) = 1\). Generally speaking, classical fields, such as thermal light and coherent light, prefer to distribute themselves in bunches rather than at random. They exhibit less correlation for time longer than the correlation time. This is the so-called bunching effect\[^{[16]}\]. On the contrary, in certain quantum optical systems, fewer quantum photons are detected close together than further apart. And the photon antibunching observed in fluorescent light from a two-level atom\[^{[19]}\] is of such kind. Here, since the two fields involved are two independent modes, we expect photons to be neither bunching nor antibunching, regardless of quantum mechanical fields or classical fields. However, as shown in Figure 3, when the Ising chain is at the critical point, the two independent fields initially
that the two initially independent quantum fields display
the classical effect due to their common interaction with
the quantum critical system. As illustrated in eq. (20),
two initially identical states evolve under two slightly
different Hamiltonians. Although the differences be-
 tween these Hamiltonians are tiny, their evolution tra-
jectories are quite distinct in the vicinity of the QPT.
Thus, this slight difference leads to the exponential de-

Figure 2 The decoherence factor $|\rho_{12}(t)|^2$ for both fields in
$(|0\rangle+|1\rangle)/\sqrt{2}$ is plotted with $\lambda = 8000$. The blue dashed line for $\lambda = 1$,
the red solid line for $\lambda = 0.1$, and the green dotted line for $\lambda = 2$. In all
figures, $t$ is in units of $1/B$.

in $(|0\rangle+|1\rangle)/\sqrt{2}$ display the photon bunching effect.
Further witness is also demonstrated in Figures 4(a)–
(c). It can also be proven that $g^{(2)}(t) < g^{(2)}(0)$ for both
fields in the coherent state $|\alpha\rangle$ which is not shown
here. In Figure 4(d), we plot the time evolution of the
second order of coherence for this case. Here, we remark

Figure 3 The second order degree of coherence $g^{(2)}(t)$ for $\lambda = 4000$ and
$(|0\rangle+|1\rangle)/\sqrt{2}$ is plotted with (a) $\lambda = 1$, (b) $\lambda = 0.1$, (c) $\lambda = 2$.

Figure 4 The second order degree of coherence $g^{(2)}(t)$ is plotted at the critical point for
$(|0\rangle+|1\rangle)/\sqrt{2}$ with (a) $N = 2000$, (b) $N = 4000$, (c) $N = 8000$. For
(d), both fields are in the coherent $|\alpha\rangle$ with $\alpha = 1$ and $N = 8000$. Note that at the steady state $g^{(2)}(t)$ is a little smaller than its original value 1 as indicated
by the red horizontal line.
decay of their decoherence factor. It can be understood as a signature of quantum chaos.  

Furthermore, for the parameters mentioned after eq. (9) and $N_c = N/10$, both the real and imaginary parts of $r_{0,1}^{(1,0)}(t) e^{i(\alpha_0 - \alpha_1)t}$ decay with a rate of the order $\sqrt{\gamma} \approx 2.5$ GHz. Since the dissipation rate of the first excitation mode is about 6.3 MHz, we can neglect the influence due to the dissipation of TLR.

### 3 Conclusion and remark

To conclude, we have explored the possibility to probe quantum criticality in the ICF by detecting the higher order quantum coherence of the two modes of cavity fields coupled to the spins. We suggest a physical implementation of this theoretical scheme based on a circuit QED system where the capacitively coupled CPBs are coupled to the TLR. Situated at the antinodes of both modes propagating in the TLR, CPBs are only coupled to the magnetic fields. In a heuristic way, we show the decoherence factor decays exponentially with time in the vicinity of the critical point. The second order of coherence is smaller than the one at the steady state. Thus, the two initially independent modes demonstrate photon bunching effect. This can serve as a witness of the QPT.

On the other hand, we have not investigated decoherence originated from the dissipation of the CPBs. We notice that in a recent work, the QPT in the dissipative random transverse-field Ising chain was investigated. It was discovered that the quantum critical point was ruined by the interplay between quantum fluctuations and Ohmic dissipation. Further exploration may be done when such kind of effect is considered.

### Appendix A Decoherence factor

Following the method introduced in ref. [14], the decoherence factor $r_{m,n'}^{(m,n)}(t)$ can be calculated in the following way.

By introducing the spin-1 pseudospin operators,  

\[ s_{zk} = \frac{1}{2}(Y_{-k} Y_k^\dagger + Y_{-k}^\dagger Y_k), \]
\[ s_{zk'} = Y_{-k} Y_k^\dagger - Y_{-k}^\dagger Y_k, \]
\[ s_{zk} = Y_{-k} Y_k^\dagger + Y_{-k}^\dagger Y_k - 1, \]

the Hamiltonian $H_0$ can also be rewritten as

\[ H_0 = \sum_{k > 0} \epsilon_k s_{zk}. \]

Because there is no energy exchange between the two modes and the qubit array, the total Hamiltonian can be decomposed into invariant subspaces with respect to the Fock state of the fields, i.e., $H = \sum_{m,n} H^{(m,n)} |m\rangle \langle n| |m\rangle$, where

\[ H^{(m,n)} = B \sum_{j=1}^N \lambda_j \sigma_j^+ \sigma_j^- + \delta \sigma_j^y \sigma_{j+1}^y. \]  

With the pseudospin operators, we can also diagonalize the Hamiltonian as

\[ H^{(m,n)} = \sum_{k>0} \alpha_k^{(m,n)} s_{zk} \]  

where $s_{zk} = s_{zk} \cos 2\alpha_k^{(m,n)} + s_{zk'} \sin 2\alpha_k^{(m,n)}$, with $2\alpha_k^{(m,n)} = \theta_k^{(m,n)} - \theta_k$, $\alpha_k^{(m,n)} = \frac{1}{2} (\lambda_{m,n} - \theta_k)$, $\theta_k^{(m,n)} = \theta_k (\lambda_{m,n})$.

Therefore, the ground state of $H_0$ is the product state of all pseudospins down $\left| - \right\rangle_k$,

\[ |G\rangle = \prod_{k>0} \left| - \right\rangle_k \]
\[ = \prod_{k>0} [\cos \alpha_k^{(m,n)} \left| - \right\rangle_k + s_{zk} \sin \alpha_k^{(m,n)} \left| + \right\rangle_k] \]  

with $\left| \pm \right\rangle_k$ being the eigen states of $s_{zk}$. Since

\[ s_k^{(m,n)} = s_{zk} \cos 2\alpha_k^{(m,n)} + s_{zk'} \sin 2\alpha_k^{(m,n)} \]  

we have

\[ r_{m,n'}^{(m,n)}(t) = \prod_{k>0} \sum_{a_k b_k} \epsilon_k^{(m,n,m',n')} e^{i(\alpha_k^{(m,n)} a_k + \epsilon_k^{(m',n')} b_k)} \]  

with

\[ C_{+,+,k} = \sin \alpha_k^{(m,n)} \cos \alpha_k^{(m',n')} \sin (\alpha_k^{(m,n)} - \alpha_k^{(m',n')}), \]
\[ C_{+,-,k} = -\cos \alpha_k^{(m,n)} \sin \alpha_k^{(m',n')} \sin (\alpha_k^{(m,n)} - \alpha_k^{(m',n')}), \]
\[ C_{-,+,k} = \sin \alpha_k^{(m,n)} \sin \alpha_k^{(m',n')} \cos (\alpha_k^{(m,n)} - \alpha_k^{(m',n')}), \]
\[ C_{-,-,k} = \cos \alpha_k^{(m,n)} \cos \alpha_k^{(m',n')} \cos (\alpha_k^{(m,n)} - \alpha_k^{(m',n')}). \]  

For a heuristic analysis, we obtain the short time behavior of $\left| r_{0,1}^{(1,0)}(t) \right|^2$ at the critical point.

\[ \left| r_{0,1}^{(1,0)}(t) \right|^2 = \prod_{k>0} F_k \]
\[ \leq \prod_{k>0} (\sin^2 (\alpha_k^{(1,0)} - \alpha_k^{(0,1)}) \cos (\epsilon_k^{(1,0)} + \epsilon_k^{(0,1)}) t). \]
$$+$ \cos^2(\alpha_{k}^{(1,0)} - \alpha_{k}^{(0,1)}) \cos(\varepsilon_{k}^{(1,0)} - \varepsilon_{k}^{(0,1)}) t$$

$$+ \sin(\alpha_{k}^{(1,0)} + \alpha_{k}^{(0,1)}) \sin(\alpha_{k}^{(1,0)} - \alpha_{k}^{(0,1)}) \cos(\varepsilon_{k}^{(1,0)} + \varepsilon_{k}^{(0,1)}) t$$

$$- \cos(\alpha_{k}^{(1,0)} + \alpha_{k}^{(0,1)}) \cos(\alpha_{k}^{(1,0)} - \alpha_{k}^{(0,1)}) \sin(\varepsilon_{k}^{(1,0)} + \varepsilon_{k}^{(0,1)}) t^2.$$

(A9)

Since all factors of $F_k$ of $\left| \varepsilon_{0,1}^{(1,0)}(t) \right|^2$ have a norm less than unity, we may expect the $\left| \varepsilon_{0,1}^{(1,0)}(t) \right|^2$ to vanish under certain conditions. Here, we set a cutoff frequency $\lambda_k$ and hence we have $\left| \varepsilon_{0,1}^{(1,0)}(t) \right|^2 \leq \prod_{k=0}^{k-1} F_k$. For small $k$, we have

$$\varepsilon_{k}^{(1,0)} \approx 2B \left| \lambda_{k,0} \right|,$$

$$\varepsilon_{k}^{(0,1)} \approx 2B \left| \lambda_{k,0} \right|,$$

$$\theta_k(\lambda) \approx \frac{k}{\lambda - 1},$$

$$\alpha_{k}^{(1,0)} \approx \frac{1}{2} \left( \frac{k}{\lambda_{k,0} - 1} - \frac{k}{\lambda_{k,0} - 1} \right),$$

$$\alpha_{k}^{(0,1)} \approx \frac{1}{2} \left( \frac{k}{\lambda_{k,0} - 1} - \frac{k}{\lambda_{k,0} - 1} \right).$$

To the second order of $\alpha_{k}^{(0,1)}$, we obtain

$$\left| \varepsilon_{0,1}^{(1,0)}(t) \right|^2 \leq \prod_{k=0}^{k-1} \left[ 1 - 2(\alpha_{k}^{(1,0)} - \alpha_{k}^{(0,1)})^2 \cos^2(\varepsilon_{k}^{(1,0)} - \varepsilon_{k}^{(0,1)}) t \right. + 2(\alpha_{k}^{(1,0)} - \alpha_{k}^{(0,1)})^2 \cos(\varepsilon_{k}^{(1,0)} + \varepsilon_{k}^{(0,1)}) t \cos(\varepsilon_{k}^{(1,0)} - \varepsilon_{k}^{(0,1)}) t$$

$$\left. + [1 - 2(\alpha_{k}^{(1,0)})^2 - (\alpha_{k}^{(0,1)})^2] \sin^2(\alpha_{k}^{(1,0)} - \alpha_{k}^{(0,1)}) t \right.$$ 

$$- 2(\alpha_{k}^{(1,0)})^2 - (\alpha_{k}^{(0,1)})^2 \sin(\varepsilon_{k}^{(1,0)} + \varepsilon_{k}^{(0,1)}) t$$

$$\times \sin(\varepsilon_{k}^{(1,0)} - \varepsilon_{k}^{(0,1)}) t.$$

Since $0 \approx \varepsilon_{k}^{(1,0)} - \varepsilon_{k}^{(0,1)} \ll \varepsilon_{k}^{(1,0)} + \varepsilon_{k}^{(0,1)} = 2\varepsilon_{k}^{(1,0)}$, we focus on the short time behavior and therefore

$$\left| \varepsilon_{0,1}^{(1,0)}(t) \right|^2 \leq \prod_{k=0}^{k-1} \left[ 1 - 2(\alpha_{k}^{(1,0)} - \alpha_{k}^{(0,1)})^2 \right]$$

$$+ 2(\alpha_{k}^{(1,0)} - \alpha_{k}^{(0,1)})^2 \cos(2\varepsilon_{k}^{(1,0)} t)$$

$$= \prod_{k=0}^{k-1} \left[ 1 - 2(\alpha_{k}^{(1,0)} - \alpha_{k}^{(0,1)})^2 \sin(2\alpha_{k}^{(1,0)} t) \right]$$

$$= \prod_{k=0}^{k-1} \left[ 1 - 2 \frac{k^2 (\lambda_{k,0} - \lambda_{k,0})}{(1 - \lambda_{k,0}) (1 - \lambda_{k,0})} \sin(2B t |1 - \lambda_{k,0}|) \right].$$

As $\lambda_{k,0} \to 1$, we have

$$\left| \varepsilon_{0,1}^{(1,0)}(t) \right|^2 \approx e^{-\gamma t},$$

where $\gamma = 4B^2 (\lambda_{0,1} - \lambda_{0,1})^2 E(k)/(|\lambda_{0,1} - 1|)^2$, $E(k) = 4\pi^2 N_c (N_c + 1)(2N_c + 1)/6N^2$ with $N_c$ being the nearest integer to $Nk_c/2\pi$.

**Appendix B Validity of rotating wave approximation**

In this section, the validity of the RWA is proven for its application in obtaining eq. (10).

The original Hamiltonian before the RWA is

$$H = E_j \sum_j \cos[\eta_j (a_j^+ + a_j) + \eta_j (a_j^+ + a_j)] \sigma_j^x$$

$$+ \frac{e^2}{C^2_{\alpha}} \sum_j \sigma_j^x \sigma_j^y \sigma_j^z.$$ 

Since $\eta_{1,2} \ll 1$, the Hamiltonian is approximated to the second order as

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2$$

$$+ E_j \sum_j \left[ 1 - \frac{1}{2} \left[ \eta_j (a_j^+ + a_j) + \eta_j (a_j^+ + a_j) \right] \right] \sigma_j^x$$

$$+ \frac{e^2}{C^2_{\alpha}} \sum_j \sigma_j^x \sigma_j^y \sigma_j^z.$$ 

(B2)

In the interaction picture with respect to $H_0 = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2$,

$$H = E_j \sum_j \left[ 1 - \frac{1}{2} \left[ \eta_j (a_j^+ + a_j) + \eta_j (a_j^+ + a_j) \right] \right] \sigma_j^x$$

$$+ \frac{e^2}{C^2_{\alpha}} \sum_j \sigma_j^x \sigma_j^y \sigma_j^z.$$ 

(B3)

Since the system evolution is determined by the time-dependent Schrödinger equation

$$H |\psi(t)\rangle = i \hbar \frac{\partial}{\partial t} |\psi(t)\rangle,$$

its solution is formally written as

$$\int_0^t |H |\psi(t)\rangle \rangle dt = i \hbar \int \left[ |\psi(t)\rangle - |\psi(0)\rangle \right].$$

(B5)

As long as $2\omega_j \gg E_j \eta_j^2 / 2$, $2\omega_j \gg E_j \eta_j^2 / 2$, $\omega_1 \pm \omega_2 \gg E_j \eta_j \eta_j$, the fast oscillating terms including the following factors $\exp[\pm i (\omega_1 \pm \omega_2) t]$, $\exp[\pm 2i \omega_1 t]$, $\exp[\pm 2i \omega_2 t]$ can be dropped for their influences are averaged out in the long run. Fortunately, for parameters listed in our paper, such requirements are fulfilled. Thus, the effec-
The effective Hamiltonian is

\[ H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + E_1 \sum_j \left[ 1 - \frac{1}{2} \eta_1^2 (2a_1^+ a_1 + 1) + \eta_2^2 (2a_2^+ a_2 + 1) \right] \sigma_j^x + \frac{e^2 C_S}{C_m} \sum_j \sigma_j^x \sigma_j^y, \]

which is exactly eq. (10).

Notice that in the above deduction we have used Riemann-Lebesgue lemma\(^\text{[22]}\):

\[
\lim_{\omega \to 0} \int_a^b f(t) e^{i\omega t} dt = \lim_{\omega \to 0} \frac{1}{i\omega} \int_a^b f(t) e^{i\omega t} dt = \lim_{\omega \to 0} \frac{1}{i} \int_a^b \frac{f(t) e^{i\omega t}}{\omega} - \frac{\omega}{\omega} \int_a^b e^{i\omega t} df(t) = 0.
\]

One (A i Q.) of the authors thanks W. Y. Huo for warm discussions.

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