An exponential spectral method using VP means for semilinear subdiffusion equations with rough data

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1. The semilinear subdiffusion equation

$$\begin{cases} \partial_t^\alpha u(x,t) - \Delta u(x,t) = f(u(x,t),x,t) & \text{for } (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases}$$

- $f \in C^k(\mathbb{R})$ is a given smooth nonlinear function
- $u_0 \in L^\infty(\Omega)$ is a given bounded measurable function
- $\partial_t^\alpha u$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$

∗ The computational challenge: u and $f(u)$ may be singular at $t = 0$.

∗ The accuracy of the numerical solution depends on the regularity/singularity.

∗ The numerical method should address the singularity of the solution.

2. The linear subdiffusion equation

- u_0 and f are smooth + compatibility conditions $\Longrightarrow$ smooth solutions

  [1] Z.-Z. Sun & X. Wu 2006

  [2] Y. Lin & C. Xu 2007

  [3] X. Li and C. Xu 2009

  [4] K. Mustapha, B. Abdallah, & K. M. Furati 2014

  [5] C. Lv & C. Xu 2016

- $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $\partial_t^m f \in L^\infty(0, T; L^2(\Omega))$ $\Longrightarrow$

$$\|\partial_t^m (u(\cdot, t) - u_0)\|_{L^2} \leq C_m t^{\alpha - m} \quad \text{for } m \geq 0.$$

  * L1, L2, dG and convolution quadrature (CQ) with a uniform stepsize generally have first-order convergence:

    [4] B. Jin, R. Lazarov & Z. Zhou 2015
    [5] W. McLean & K. Mustapha 2015
    [6] Y. Xing & Y. Yan 2018

* Optimal-order convergence of L1 and L2 schemes by graded stepsizes:

  [7] M. Stynes, E. O'Riordan & J. L. Gracia 2017
  [8] N. Kopteva & X. Meng 2020
  [9] N. Kopteva 2021

* High-order convergence by dG under the additional regularity condition

$$\partial_t u \in L^2(0,T;H^2(\Omega)) :$$

  (which basically requires $u_0 \in H^{5/2}(\Omega) \cap H_0^1(\Omega) + \Delta u_0 = 0$ on $\partial\Omega$)
  [10] K. Mustapha, B. Abdallah & K. M. Furati 2014

- $u_0 \in L^p(\Omega)$ and $\partial_t^m f \in L^\infty(0,T;L^p(\Omega)) \implies$

$$\|\partial_t^m u(\cdot,t)\|_{L^p} \le C_m t^{-m} \quad \text{for } m \ge 0.$$

  * Convolution quadrature for linear subdiffusion equation:

$$\|u(\cdot,t_n) - u_n\|_{L^p} \le C t_n^{-k} \tau^k$$

[11] C. Lubich, I. H. Sloan & V. Thomée 1996 $(k = 1,2)$
[12] B. Jin, B. Li & Z. Zhou 2017 $(k = 1,\dots,6)$
[13] Y. Xing & Y. Yan 2018 $(k = 2-\alpha, 3-\alpha)$

3. The semilinear subdiffusion equation

Additional computational challenge: $f(u)$ is also singular at $t = 0$.

[14] B. Jin, B. Li & Z. Zhou 2018: $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and

$$\|u(\cdot, t_n) - u_n\|_{L^p} \leq C\tau^\alpha$$

[15] M. Al-Maskari & S. Karaa 2019 : $u_0 \in L^2(\Omega)$ and

$$\|u(\cdot, t_n) - u_n\|_{L^2} \leq Ct_n^{-1}\tau$$

[16] K. Wang & Z. Zhou 2020: $u_0 \in H_0^1(\Omega) \cap C^2(\overline{\Omega})$ and

$$\|u(\cdot, t_n) - u_n\|_{L^p} \leq Ct_n^{-1-\alpha+\epsilon}\tau^{1+2\alpha-\epsilon}$$

[17] B. Li & S. Ma 2022: $u_0 \in L^\infty(\Omega)$ and

$$\|u(\cdot, t_n) - u_n\|_{L^p} \leq C\tau^k$$

Discrete Gronwall's inequality:

[18] B. Jin, B. Li & Z. Zhou 2018 (uniform stepsize)

[19] Li, Liao, Sun, Wang & Zhang 2018 (uniform stepsize)

[20] H.-L. Liao, W. McLean & J. Zhang. 2019 (graded stepsizes)

4. Spectral methods for the subdiffusion equation

[21] X. Li & C. Xu 2009: polynomial projection (for smooth solutions).

[22] F. Chen, Q. Xu & J. S. Hesthaven 2015: (stability analysis).

[23] Zayernouri & Karniadakis 2013, 2014: generalized Jacobi functions (numerical method for fractional ODEs and PDEs)

[24] S. Chen, J. Shen & L. Wang 2016: generalized Jacobi functions (numerical analysis for fractional ODEs)

[25] S. Chen, J. Shen & L. Wang 2016: generalized Jacobi functions (numerical analysis for fractional ODEs)

[26] X. Zhao & Z. Zhang 2016: generalized Jacobi functions (superconvergence of spectral interpolation)

[27] D. Hou, M. T. Hasan & C. Xu, 2017-2018: Müntz spectral methods.

$$t^{-1+m}\partial_t^m[u(x,t^{1/\lambda})] \in L^2(0,T;H_0^1(\Omega)) \quad \forall\, m \geq 0.$$

- This covers a wide class of solutions, including solutions in the form of

$$u(x,t) = \sum_{j=1}^{\infty} t^{j\alpha}\phi_j(x) \quad \text{(choose } \lambda = 1/\alpha \text{ in the algorithm).}$$

- In general, if $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ then

$$\|\partial_t^m(u(\cdot,t)-u_0)\|_{L^2} \leq C_m t^{\alpha-m} \quad \text{for } m \geq 0.$$

A fixed high-order convergence by choosing a sufficiently small λ.

[28] S. Chen & J. Shen 2022: Log orthogonal functions (approximation properties).

[29] S. Chen, J. Shen, Z. Zhang, and Z. Zhou 2020: Log orthogonal functions (for the subidffusion equation with $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$, $\Delta u_0 = 0$ on $\partial\Omega$)

Current error analyses:

- Linear subdiffusion equation
- $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ + some compatibility conditions
- Regularity condition:

$$\|\partial_t^m (u(\cdot,t) - u_0)\|_{L^2} \le C_m t^{\alpha-m} \quad \text{for } m \ge 0.$$

- Hilbert space framework.

Open and challenging questions (to be addressed in this paper):

- L^∞ initial data
- Regularity condition:

$$\|\partial_t^m u(\cdot,t)\|_{L^\infty} + \|\partial_t^m f(u(\cdot,t),\cdot,t)\|_{L^\infty} \le C_m t^{-m} \quad \text{for } m \ge 0$$

- L^∞-norm based Banach space framework

5. An exponential spectral method

$$\begin{cases} \partial_t^\alpha (u - u_0) - \Delta u = f(u) & \text{in } \Omega \times (0, T] \\ u = 0 & \text{on } \partial\Omega \times (0, T] \\ u(0) = u_0 & \text{in } \Omega \end{cases} \qquad \text{with } 0 < \alpha < 1$$

Laplace transform in time:

$$z^\alpha \hat{u}(z) - z^{\alpha-1} u_0 - \Delta \hat{u}(z) = \hat{g}(z), \qquad \text{where } g(t) = f(u(t))$$

$$\Longrightarrow \quad \hat{u}(z) = (z^\alpha - \Delta)^{-1} z^{\alpha-1} u_0 + (z^\alpha - \Delta)^{-1} \hat{g}(z)$$

Inverse Laplace transform:

$$\begin{aligned} u(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}(z)=\sigma} e^{zt} (z^\alpha - \Delta)^{-1} z^{\alpha-1} u_0 \mathrm{d}z \\ + \frac{1}{2\pi i} \int_{\mathrm{Re}(z)=\sigma} e^{zt} (z^\alpha - \Delta)^{-1} \widehat{g}(z) \mathrm{d}z \end{aligned}$$

$$\begin{aligned}
u(t) &= \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=\sigma} e^{zt}(z^\alpha-\Delta)^{-1}z^{\alpha-1}u_0\mathrm{d}z + \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=\sigma} e^{zt}(z^\alpha-\Delta)^{-1}\widehat{g}(z)\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=\sigma} e^{zt}(z^\alpha-\Delta)^{-1}z^{\alpha-1}u_0\mathrm{d}z + \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=\sigma} (z^\alpha-\Delta)^{-1}\int_0^t e^{z(t-s)}g(s)\mathrm{d}s\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=\sigma} e^{zt}(z^\alpha-\Delta)^{-1}z^{\alpha-1}u_0\mathrm{d}z + \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=\sigma} (z^\alpha-\Delta)^{-1}\int_0^t e^{z(t-s)}f(u(s))\mathrm{d}s\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=\sigma} e^{zt}(z^\alpha-\Delta)^{-1}z^{\alpha-1}u_0\mathrm{d}z + \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=\sigma} (z^\alpha-\Delta)^{-1}y(z,t)\mathrm{d}z
\end{aligned}$$

where $y(z,t) = \int_0^t e^{z(t-s)}f(u(s))\mathrm{d}s$, which satisfies the ordinary differential equation:

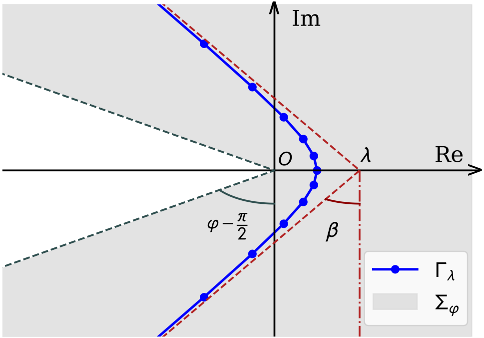
$$\frac{\mathrm{d}y(z,t)}{\mathrm{d}t} = zy(z,t) + f(u(t))$$

Deformation of contour:

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_\lambda} e^{zt}(z^\alpha - \Delta)^{-1} z^{\alpha-1} u_0 \mathrm{d}z + \frac{1}{2\pi i}\int_{\Gamma_\lambda} (z^\alpha - \Delta)^{-1} y(z,t) \mathrm{d}z$$

where $y(z,t)$ satisfies the ordinary differential equation:

$$\frac{\mathrm{d}y(z,t)}{\mathrm{d}t} = zy(z,t) + f(u(t)) \quad \text{with} \quad y(z,0) = 0.$$



For $t \in [t_{n-1}, t_n]$, we have

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_\lambda} e^{zt}(z^\alpha - \Delta)^{-1} z^{\alpha-1} u_0 \mathrm{d}z + \frac{1}{2\pi i}\int_{\Gamma_\lambda} e^{zt}(z^\alpha - \Delta)^{-1} y(z,t)\mathrm{d}z$$
$$= \sum_{j=-M}^{M} \omega_j e^{z_j} z_j^{\alpha-1}(z_j^\alpha - t^\alpha \Delta)^{-1} u_0 + \sum_{j=-M}^{M} \tilde{\omega}_j (\tilde{z}_j^\alpha - \Delta)^{-1} y(\tilde{z}_j, t) + \mathcal{E}_q(t)$$

Quadrature error:

$$\|\mathcal{E}_q(t)\|_{L^\infty} \le C e^{-\sqrt{M/C}}$$

Duhamel formula: For $t \in [t_{n-1}, t_n]$, we have

$$y(z,t) = e^{z(t-t_{n-1})} y(z, t_{n-1}) + \int_{t_{n-1}}^{t} e^{z(t-s)} f(u(s))\mathrm{d}s$$
$$= e^{z(t-t_{n-1})} y(z, t_{n-1}) + \int_{t_{n-1}}^{t} e^{z(t-s)} V_m^r f(u(s))\mathrm{d}s + \mathcal{E}_f^n(t_n)$$
$$= e^{z(t-t_{n-1})} y(z, t_{n-1}) + \int_{t_{n-1}}^{t} e^{z(t-s)} \sum_{i=1}^{m} f(u(t_{m,i}))\Phi_{m,i}^r(s)\mathrm{d}s + \mathcal{E}_f^n(t)$$

5. An exponential spectral method

VP means on the standard interval $[-1,1]$:

$$V_m^r g(t) = \sum_{i=1}^{m} g(t_{m,i})\Phi_{m,i}^r(t) \quad \text{for } g \in C([-1,1]),$$

where

$$\Phi_{m,i}^r(t) = \frac{\sum_{j=0}^{m+r-1} \mu_{m,j}^r J_j^{\alpha,\beta}(t_{m,i}) J_j^{\alpha,\beta}(t)}{\sum_{j=0}^{m-1} [J_j^{\alpha,\beta}(t_{m,i})]^2}, \;\; j = 1, \dots m,$$

Properties of VP means:

$$\|V_m^r g\|_{C([-1,1];L^\infty(\Omega))} \le C \sup_{1\le i\le m} \|g(t_{m,i})\|_{L^\infty(\Omega)}$$

$$\|g - V_m^r g\|_{C([-1,1];L^\infty(\Omega))} \le C_k m^{-k} \|\partial_t^k g\|_{C([-1,1];L^\infty(\Omega))}$$

where C does not depend on m.

Interpolation on a scaled interval $[t_{j-1}, t_j]$: For every fixed $k \ge 1$,

$$\begin{aligned}\|f(u) - V_m^r f(u)\|_{C([t_{j-1},t_j];L^\infty(\Omega))} &\le C_k m^{-k} \left(\frac{t_j - t_{j-1}}{2}\right)^k \|\partial_t^k f(u)\|_{C([t_{j-1},t_j];L^\infty(\Omega))} \\ &\le C_k m^{-k} \left(\frac{t_j - t_{j-1}}{2}\right)^k t_{j-1}^{-k} \quad (\text{if } u_0 \in L^\infty(\Omega))\end{aligned}$$

Choose internal nodes $t_{n-1} < t_{n1} < t_{n2} < \cdots < t_{nm} < t_n$ and consider the collocation method:

$$U(t_{ni}) = \sum_{j=-M}^{M} \omega_j e^{z_j} z_j^{\alpha-1}(z_j^\alpha - t_{ni}^\alpha \Delta)^{-1} u_0 + \sum_{j=-M}^{M} \tilde{\omega}_j (\tilde{z}_j^\alpha - \Delta)^{-1} Y(\tilde{z}_j, t_{ni})$$

$$Y(z, t_{ni}) = e^{z(t_{ni}-t_{n-1})} Y(z, t_{n-1}) + \int_{t_{n-1}}^{t_{ni}} e^{z(t_{ni}-s)} \sum_{i=1}^{m} f(U(t_{m,i})) \Phi_{m,i}^r(s) \mathrm{d}s$$

- On $[t_{n-1}, t_n]$, we solve the nonlinear system collocating at the internal nodes t_{ni}.
- $[t_{n-1}, t_n] = [\lambda^{n-1-N(m)} T, \lambda^{n-N(m)} T]$ for some $\lambda > 1$, for $n = 1, \ldots, N(m)$, where $N = N(m)$ satisfies
$$\lim_{m\to\infty} \frac{N(m)}{\log(m)} = \infty,$$

6. Existence and uniqueness of numerical solutions

We first consider the problem on $[0, T]$, and define a map $V(t_{ni}) = G(U(t_{ni}))$ by

$$V(t_{ni}) = \sum_{j=-M}^{M} \omega_j e^{z_j} z_j^{\alpha-1} (z_j^\alpha - t_{ni}^\alpha \Delta)^{-1} u_0 + \sum_{j=-M}^{M} \tilde{\omega}_j (\tilde{z}_j^\alpha - \Delta)^{-1} Y(\tilde{z}_j, t_{ni})$$

$$Y(z, t_{ni}) = e^{z(t_{ni} - t_{n-1})} Y(z, t_{n-1}) + \int_{t_{n-1}}^{t_{ni}} e^{z(t_{ni} - s)} \sum_{i=1}^{m} f(U(t_{m,i})) \Phi_{m,i}^r(s) \mathrm{d}s$$

- If T is smaller than some constant, then $G : L^\infty(\Omega)^m \to L^\infty(\Omega)^m$ is a contraction and therefore has a unique fixed point, i.e., a numerical solution of the proposed method.
- If T is not small then we consider the problem on $[0, T_0]$ and $[T_0, T]$ separately, with T_0 being a small number. On $[T_0, T]$ the solution is sufficiently smooth and therefore the problem is easier.

7. Error estimates

Exact solution u:

$$\begin{cases} \partial_t^\alpha u - \Delta u = V_m^r f(u) + \mathcal{E}_f & \text{in } \Omega \times (0,T], \\ u = 0 & \text{on } \partial\Omega \times (0,T], \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

Auxiliary function u^*:

$$\begin{cases} \partial_t^\alpha u^* - \Delta u^* = V_m^r f(U) & \text{in } \Omega \times (0,T], \\ u^* = 0 & \text{on } \partial\Omega \times (0,T], \\ u^*(0) = u_0 & \text{in } \Omega. \end{cases}$$

Expressions of the solution:

$$u(t) = F(t)u_0 + \int_0^t E(t-s)[V_m^r f(u)](s)\mathrm{d}s + \int_0^t E(t-s)\mathcal{E}_f(s)\mathrm{d}s$$

$$u^*(t) = F(t)u_0 + \int_0^t E(t-s)[V_m^r f(U)](s)\mathrm{d}s$$

where

$$E(t) = \frac{1}{2\pi i}\int_{\Gamma_\lambda} e^{zt}(z^\alpha - \Delta)^{-1}\mathrm{d}z, \quad \|E(t)\|_{L^\infty \to L^\infty} \le C(t-s)^{\alpha-1}.$$

7. Error estimates

Expression of the error $\tilde{e} = u - u^*$:

$$\tilde{e}(t) = \int_0^t E(t-s)[V_m^r f(u) - V_m^r f(U)](s)\mathrm{d}s + \int_0^t E(t-s)\mathcal{E}_f(s)\mathrm{d}s$$

L^∞ norm of the error $\tilde{e}$:

$$\begin{aligned}\|\tilde{e}(t_{ni})\|_{L^\infty(\Omega)} \le C &\int_0^{t_{ni}} (t_{ni}-s)^{\alpha-1} \max_{1\le l\le m} \|e(t_{jl})\|_{L^\infty(\Omega)}\mathrm{d}s \\ &+ \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} C_k (t_{ni}-s)^{\alpha-1} m^{-k} \Big(\frac{t_j - t_{j-1}}{2}\Big)^k t_{j-1}^{-k}\mathrm{d}s \\ &+ \int_0^{t_1} C_k (t_{ni}-s)^{\alpha-1} \|f(u) - V_m^r f(u)\|_{L^\infty(\Omega)}\mathrm{d}s\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}\max_{n,i} \|\tilde{e}(t_{ni})\|_{L^\infty(\Omega)} &\le CT^\alpha \max_{n,i} \|e(t_{ni})\|_{L^\infty(\Omega)} + C_k m^{-k} \sum_{j=2}^{n} \lambda^{[(1-\gamma)+\alpha](j-N)} + Ct_1^\alpha \\ &\le CT^\alpha \max_{n,i} \|e(t_{ni})\|_{L^\infty(\Omega)} + Cm^{-k} + C\lambda^{-N\alpha} \\ &\le CT^\alpha \max_{n,i} \|e(t_{ni})\|_{L^\infty(\Omega)} + Cm^{-k} \quad \Big(\lim_{m\to\infty} \frac{N(m)}{\log(m)} = \infty\Big)\end{aligned}$$

7. Error estimates

Since

$$u^*(t_{ni}) = F(t)u_0 + \int_0^{t_{ni}} E(t_{ni}-s)[V_m^r f(U)](s)\mathrm{d}s$$
$$= \frac{1}{2\pi i}\int_{\Gamma_\lambda} e^z (z^\alpha - t_{ni}^\alpha \Delta)^{-1} z^{\alpha-1} u_0 \mathrm{d}z + \frac{1}{2\pi i}\int_{\Gamma_\lambda} e^{zt_{ni}} (z^\alpha - \Delta)^{-1} Y(z, t_{ni})\mathrm{d}z$$

$$U(t_{ni}) = \sum_{j=-M}^{M} \omega_j e^{z_j} z_j^{\alpha-1}(z_j^\alpha - t_{ni}^\alpha \Delta)^{-1} u_0 + \sum_{j=-M}^{M} \tilde{\omega}_j (\tilde{z}_j^\alpha - \Delta)^{-1} Y(\tilde{z}_j, t_{ni}),$$

it follows that

$$\max_{n,i} \|u^*(t_{ni}) - U(t_{ni})\|_{L^\infty(\Omega)} \le Ce^{-\sqrt{M/C}}.$$

L^∞ norm of the error $e(t_{ni}) = u(t_{ni}) - U(t_{ni})$:

$$\max_{n,i} \|e(t_{ni})\|_{L^\infty(\Omega)} \le CT^\alpha \max_{n,i} \|e(t_{ni})\|_{L^\infty(\Omega)} + Cm^{-k} + Ce^{-\sqrt{M/C}}$$

$\Longrightarrow$ (if T is smaller than some constant)

$$\max_{n,i} \|e(t_{ni})\|_{L^\infty(\Omega)} \le Cm^{-k} + Ce^{-\sqrt{M/C}}$$

7. Error estimates

If T is not smaller than the required constant, we can consider a division $[0,T]=[0,T_1]\cup[T_1,T_2]\cup[T_2,T_3]\cup\cdots\cup[T_{L-1},T_L]$, with T_j-T_{j-1} sufficiently small.

For $t_{ni}\in[0,T_1]$ we have already obtained the desired error estimate

$$\max_{t_{ni}\in[0,T_1]}\|e(t_{ni})\|_{L^\infty(\Omega)}\le Cm^{-k}+Ce^{-\sqrt{M/C}}$$

For $t_{ni}\in[T_1,T_2]$ we have

$$\begin{aligned}\|\tilde e(t_{ni})\|_{L^\infty(\Omega)}\le{}& C\int_{T_1}^{t_{ni}}(t_{ni}-s)^{\alpha-1}\max_{j,l}\|e(t_{jl})\|_{L^\infty(\Omega)}\mathrm{d}s\\ &+C\int_{T_1}^{t_{ni}}(t_{ni}-s)^{\alpha-1}m^{-k}\Big(\frac{t_j-t_{j-1}}{2}\Big)^k t_{j-1}^{-k}\mathrm{d}s\\ &+C_km^{-k}+Ce^{-\sqrt{M/C}}\end{aligned}$$

$\Longrightarrow$

$$\max_{n,i}\|\tilde e(t_{ni})\|_{L^\infty(\Omega)}\le C(T_2-T_1)^\alpha\max_{n,i}\|e(t_{ni})\|_{L^\infty(\Omega)}+C_km^{-k}+Ce^{-\sqrt{M/C}}$$

$\Longrightarrow$

$$\max_{n,i}\|e(t_{ni})\|_{L^\infty(\Omega)}\le C(T_2-T_1)^\alpha\max_{n,i}\|e(t_{ni})\|_{L^\infty(\Omega)}+C_km^{-k}+Ce^{-\sqrt{M/C}}$$

The estimates on $[T_2,T_3]$, $[T_3,T_4]$, ..., are similar.

Theorem: Let $u \in C([0,T];L^2(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$ be a bounded mild solution of the subdiffusion equation with initial value $u_0 \in L^\infty(\Omega)$ and nonlinear source function $f \in C^\infty(\mathbb{R})$. Then for sufficiently large m (degree of polynomials)

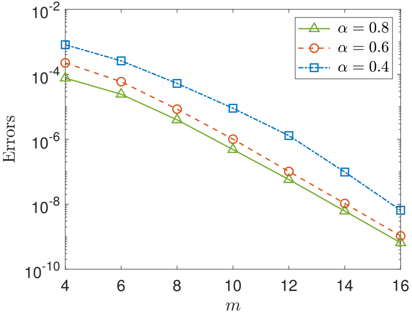
- $\exists$ a unique numerical solution in an L^∞-neighborhood of the mild solution.
- Error estimate:

$$\max_{n,i} \|u(t_{ni}) - U(t_{ni})\|_{L^\infty(\Omega)} \le C_k(m^{-k} + e^{-\sqrt{M}/C}),$$

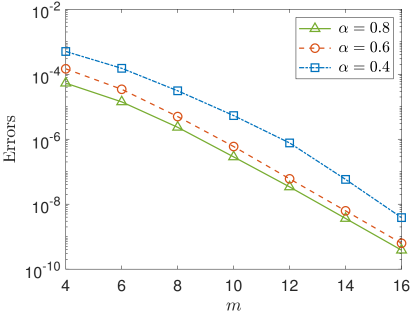
  which holds for all $1 \le k \le m$.

8. Numerical examples

Domain $\Omega = (0,1)$, $T = 1$, $u_0 = \chi_{[1/2,1)} \in L^\infty(\Omega)$.

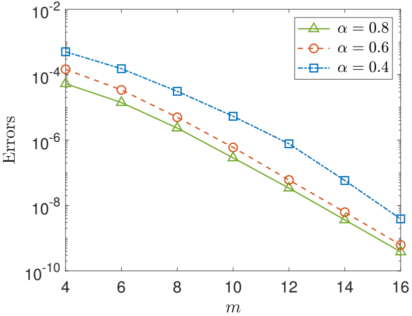


(a) $f(x,t) = \sin t \cos \pi x$

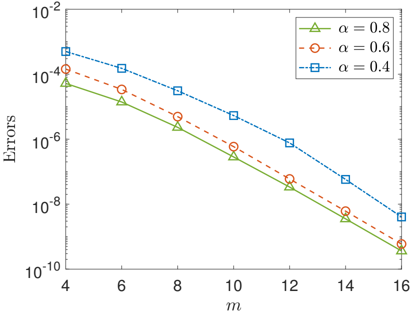


(b) $f(x,t) = t^{0.75} \cos \pi x$

Domain $\Omega = (0,1)$, $T = 1$, $u_0 = \chi_{[1/2,1)} \in L^\infty(\Omega)$.



(c) $f(x,t) = t^{0.5}\cos \pi x$



(d) $f(x,t) = t^{0.25}\cos \pi x$

Domain $\Omega = (0,1)$, $T = 1$, $u_0 = \chi_{[1/2,1)} \in L^\infty(\Omega)$.

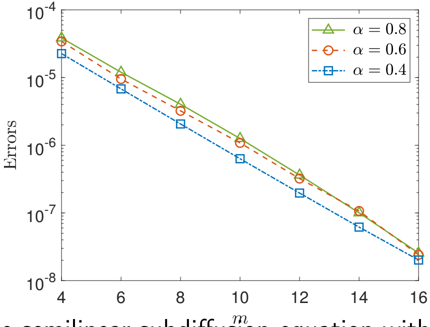


Figure: The semilinear subdiffusion equation with $f(u) = \sin u$

9. Conclusions

- A new spectral method for the linear and semilinear subdiffusion equations
- Rough initial data in $L^\infty(\Omega)$
- Regularity condition:

$$\|\partial_t^m u(\cdot,t)\|_{L^\infty} + \|\partial_t^m f(u(\cdot,t),\cdot,t)\|_{L^\infty} \le C_m t^{-m} \quad \text{for } m \ge 0$$

  which holds if $u \in C([0,T];L^2(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$.

- The combination of several techniques:
  - Contour integral representation of the mild solutions
  - Quadrature approximation of the contour integrals
  - Exponential integrator using VP means
  - Decomposition of the time interval geometrically refined towards $t = 0$
  - Spectral convergence in the L^∞-norm framework
- The nonlinear source function may not be globally Lipschitz continuous.

Open question: High-order convergence in space.

Thank you for your attention.