A quadrature scheme for steady-state diffusion equations involving fractional power of regularly accretive operator

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Target problem

We investigate numerical approach to solve the following nonlocal diffusion equation:

$$(\mathcal{A}^\alpha + b\mathcal{I})u = f, \tag{1.1}$$

where $\alpha \in (0,1)$, $b \in [1, +\infty)$, and $\mathcal{A}^\alpha$ denotes the α-th power of $\mathcal{A}$, where $\mathcal{A}$ corresponds to the following sesquilinear form: $\forall w, v \in V$

$$A(w,v) = \int_\Omega \nabla w\, \mathbf{C}(x)(\overline{\nabla v})^T + (\mathbf{a}(x)\cdot\nabla)w\bar{v} + r(x)w\bar{v}\,dx \tag{1.2}$$

where $\mathbf{C}$ is a complex-valued $d \times d$ matrix, $\mathbf{a}(x)$ is a complex-valued d-dimensional vector and $r(x)$ is a complex-valued scalar function.

Dunford-Taylor : $$\mathcal{A}^\alpha = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}'} z^\alpha (z\mathcal{I} - \mathcal{A})^{-1} d. \tag{1.3}$$

We further assume that the sesquilinear form $A(\cdot,\cdot)$ satisfies strong ellipticity

$$\Re(A(v,v)) \geq c_0\|v\|_V^2 \quad \text{for all } v \in V, \tag{1.4}$$

and continuity

$$|A(w,v)| \leq c_1\|w\|_V\|v\|_V \quad \text{for all } w, v \in V, \tag{1.5}$$

where c_0, c_1 are positive real numbers. Note that the two inequalities above also imply $c_0 \leq c_1$.

Apart from pure mathematical interest, (1.1) can be obtained from time discretization of parabolic equations involving $\mathcal{A}^\alpha$. For example, when we apply linearized backward Euler scheme to the model problem

$$\frac{\partial w(x,t)}{\partial t} + \mathcal{A}^\alpha w(x,t) = g(x,t,w), \tag{1.6}$$

we obtain

$$\left(\mathcal{A}^\alpha + \frac{1}{\Delta t}\right) w^{n+1}(x) = g(x, t^{n+1}, w^n) + \frac{w^n(x)}{\Delta t} \tag{1.7}$$

with Δt the temporal step size and $w^{n+1}(x)$ the semidiscrete solution at $t_{n+1} = (n+1)\Delta t$.

previous work

- Most work focuses on $\mathcal{A}^\alpha u = f$ with Hermitian $\mathcal{A}$, see[1] for a review;
- $\mathcal{A}^\alpha u + bu = f$ with Hermitian $\mathcal{A}$: BURA[2]; rational Krylov subspace-based techniques[3] together with contour integral proposed in [4]

[1]S. Harizanov, R. Lazarov, and S. Margenov, A survey on numerical methods for spectral space-fractional diffusion problems, Fractional Calculus and Applied Analysis, 23 (2020), pp. 1605–1646.

[2]S. Harizanov, R. Lazarov, S. Margenov, and P. Marinov, Numerical solution of fractional diffusion–reaction problems based on BURA, Computers & Mathematics with Applications, 80 (2020), pp. 316– 331.

[3]K. Burrage, N. Hale, and D. Kay, An efficient implicit FEM scheme for fractional-in-space reaction- iffusion equations, SIAM Journal on Scientific Computing, 34 (2012), pp. A2145–A2172.

[4]N. Hale, N. J. Higham, and L. N. Trefethen, Computing A^α, $\log A$, and related matrix functions by contour integrals, SIAM Journal on Numerical Analysis, 46 (2008), pp. 2505–2523.

Definitions:

Let X be a Hilbert space. A complex-valued function $\Psi(v,w)$ defined for v,w belonging to a linear subset of X is called a sesquilinear form if it is linear in v and semilinear in w.

We use $D(\Psi)$ to denote the domain of Ψ. For a given sesquilinear form $\Psi(\cdot,\cdot)$, $\Psi^*(v,w) = \overline{\Psi(w,v)}$ defines another sesquilinear form Ψ^* with $D(\Psi^*) = D(\Psi)$. We say Ψ^* is the *adjoint form* of Ψ. Ψ is said to be Hermitian or symmetric if $\Psi = \Psi^*$.

A Hermitian form Φ is said to be *nonnegative* if $\Phi(v,v) \geq 0$ for all $v \in D(\Phi)$. A nonnegative Hermitian form Φ is said to be *closed* if $v_n \in D(\Phi)$, $v_n \to v \in X$ and $\Phi(v_n - v_m, v_n - v_m) \to 0$ is sufficient to derive $v \in D(\Phi)$ and $\Phi(v_n, v_n) \to \Phi(v,v)$.

For a given sesquilinear form, one can decompose it into two Hermitian forms by

$$\Psi = \frac{1}{2}\left(\Psi + \Psi^*\right) + \mathrm{i}\frac{1}{2\mathrm{i}}\left(\Psi - \Psi^*\right) := \Psi_{\mathtt{Re}} + \mathrm{i}\Psi_{\mathtt{Im}},$$

which are referred as the *real* and *imaginary* parts of Ψ, respectively. It is worth to point out that $\Psi_{\mathtt{Re}}$ and $\Psi_{\mathtt{Im}}$ are not real-valued.

Definition 1.1

A sesquilinear form $\Psi(\cdot,\cdot)$ will be said to be regular if

1. the domain of $\Psi(\cdot,\cdot)$ is dense in X;
2. $\Psi_{\mathtt{Re}}$ is a closed, nonnegative Hermitian form;
3. there exists some $\widetilde{\beta} \geq 0$ such that $|\Psi_{\mathtt{Im}}(v,v)| \leq \widetilde{\beta}\Psi_{\mathtt{Re}}(v,v)$ holds for $\forall v \in D(\Psi)$.

An operator will be said to be regular if it is associated with a regular sesquilinear form. The smallest number $\widetilde{\beta}$ is called the index of Ψ or the index of the associated operator.

An operator $\mathcal{B}$ is said to be accretive if $\Re(\mathcal{B}v, v) \geq 0$ for $\forall v \in D(\mathcal{B})$. Further, we call $\mathcal{B}$ is maximal accretive if $\lambda + \mathcal{B}$ is surjective for $\lambda \geq 0$.

Obviously, the sesquilinear form $A(\cdot,\cdot)$ satisfies 1) and 2) in Definition (1.1) with $X = L^2(\Omega)$.
In addition, (1.4) and (1.5) imply

$$|A_{\mathtt{Im}}(v,v)| \leq \frac{\sqrt{c_1^2 - c_0^2}}{c_0} A_{\mathtt{Re}}(v,v).$$

Lemma 1.2

Suppose $\mathcal{B} : D(\mathcal{B}) \to X$ is an operator in Banach space X and its resolvent contains $\Sigma_\omega \cup \{z : |z| < \lambda_0\}$, where $\Sigma_\omega = \{z \in \mathbb{C} : \pi - |\arg z| < \omega\}$ with constant $\lambda_0 > 0$ and $\omega \in (0, \pi)$. Then the following integral formula holds for $\alpha \in (0,1)$ and $t \geq 0$

$$(\mathcal{B}^\alpha + t\mathcal{I})^{-1} = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} (\rho + \mathcal{B})^{-1} \frac{\rho^\alpha}{\rho^{2\alpha} + 2t\cos \pi\alpha \rho^\alpha + t^2} d\rho. \quad (1.8)$$

Proof: This can be done by utilizing the following integral curve :

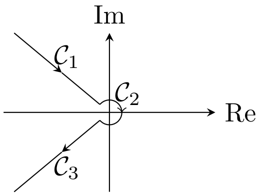

Figure 1: Integral curve on the complex plane.

Since $\mathcal{A}$ is maximal regularly accretive, $\sigma(\mathcal{A})$ is contained in the numerical range of $\mathcal{A}$, i.e.

$$\sigma(\mathcal{A}) \subset \{\mu : \mu = \frac{(\mathcal{A}v, v)}{\|v\|^2}, v \in V, v \neq 0\}.$$

Thus appealing to (1.4) we know $\mathcal{A}$ satisfies the conditions in Theorem 1.2. It is worth to point out that for $t = 0$, (1.8) reduces to Balakrishnan formula.

We denote by $\mathcal{S}$ the self-adjoint operator associate with $A_{\mathtt{Re}}$. The self-adjoint operator $\mathcal{S}$ naturally induces a Hilbert space $\dot{H}^{\mu}$, given by $\dot{H}^{\mu} = D(\mathcal{S}^{\mu/2})$. In terms of eigenfunction expansion we can characterize $\dot{H}^{\mu}$ by

$$\dot{H}^{\mu} := D(\mathcal{S}^{\mu/2}) = \left\{ \sum_{j=1}^{\infty} c_j \phi_j : \sum_{j=1}^{\infty} |c_j|^2 \lambda_j^{\mu} < \infty \right\}.$$

The corresponding inner product is

$$(w, v)_{\mu} = \sum_{j=1}^{\infty} \lambda_j^{\mu} (w, \phi_j) \overline{(v, \phi_j)}.$$

The antilinear and linear functionals on $\dot{H}^{\mu}$, say $H_a^{-\mu}$ and $H_l^{-\mu}$ are respectively given by

$$H_a^{-\mu} = \big\{\langle v, \cdot\rangle : v \in \dot{H}^{\mu}\big\}, \quad \text{and} \quad H_l^{-\mu} = \big\{\langle \cdot, v\rangle : v \in \dot{H}^{\mu}\big\}.$$

Obviously we have $V_a^* = H_a^{-1}$ and $V_l^* = H_l^{-1}$. Assume (1.4)[strong ellipticity] and (1.5)[continuity] hold, then it follows for $\mu \in [0, 1)$[5]

$$D(\mathcal{A}^{\mu/2}) = D((\mathcal{A}^*)^{\mu/2}) = D(\mathcal{S}^{\mu/2}).$$

[5] T. Kato, Fractional powers of dissipative operators, 1961

Let $H^s(\Omega)(s > 0)$ denote the Sobolev space of order s. We introduce the following spaces equipped with their natural norms:

$$\widetilde{H}^s := \begin{cases} \dot{H}^s & \text{for} \quad s \in [0,1], \\ H^s(\Omega) \cap V & \text{for} \quad s > 1. \end{cases}$$

Assumption 1.1

Assume that for $s \in [0, \gamma]$ with $\gamma \in [0, 1]$, $D(\mathcal{A}^{\frac{s+1}{2}}) \subset \widetilde{H}^{1+s}$.

$s = 0$: the famous Kato square root problem which has been intensively studied under different boundary conditions[6]

$s \in (0, \gamma]$: it can be demonstrated that the inclusion relation holds provided that γ is the elliptic regularity index of $\mathcal{A}$, that is, $T(\mathcal{A}^{-1})$ is a bounded map from H_a^{-1+s} into $\widetilde{H}^{1+s}$ for $s \in (0, \gamma]$, see [7].

[6] A. Axelsson, S. Keith, and A. McIntosh, The kato square root problem for mixed boundary value problems, Journal of the London Mathematical Society, 74 (2006), pp. 113–130.

[7] A. Bonito and J. E. Pasciak, Numerical approximation of fractional powers of regularly accretive operators, IMA Journal of Numerical Analysis, (2017), pp. 1245–1273.

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We shall investigate the error between u and u_h where

$$u = (\mathcal{A}^\alpha + b)^{-1} f, \qquad u_h = (\mathcal{A}_h^\alpha + b)^{-1} \pi_h f.$$

In fact,

$$u = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} (\rho + \mathcal{A})^{-1} f \frac{\rho^\alpha}{\rho^{2\alpha} + 2b\cos\pi\alpha\rho^\alpha + b^2} d\rho. \tag{2.1}$$

$$u_h = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} (\rho + \mathcal{A}_h)^{-1} f_h \frac{\rho^\alpha}{\rho^{2\alpha} + 2b\cos\pi\alpha\rho^\alpha + b^2} d\rho. \tag{2.2}$$

Let w_ρ^ε and $w_{\rho,h}^\varepsilon$ denote the solutions of the following equations

$$(\rho + \varepsilon\mathcal{A})w_\rho^\varepsilon = f \quad \text{and} \quad (\rho + \varepsilon\mathcal{A}_h)w_{\rho,h}^\varepsilon = f_h,$$

then we can get the following estimate:

Lemma 2.1

Suppose Assumption 1.1 holds and $f \in D(\mathcal{A}^{\frac{\delta}{2}})$ with $\delta \in [0,2]$, then we have

$$\|w_\rho^\varepsilon - w_{\rho,h}^\varepsilon\| \leq c\varepsilon^{-1}\left(h^{\sigma+\widetilde{\sigma}}\widetilde{\rho}^{\max(\frac{\sigma+\widetilde{\sigma}-\delta}{2}-1,\frac{\widetilde{\sigma}-1}{2}-1)} + h^{\sigma'+\widetilde{\sigma}}\widetilde{\rho}^{\max(\frac{\sigma'+\widetilde{\sigma}-\delta}{2}-1,\frac{\widetilde{\sigma}}{2}-1)}\right)\|\mathcal{A}^{\frac{\delta}{2}}f\| \tag{2.3}$$

where $\sigma, \widetilde{\sigma} \in [0,\gamma]$, $\sigma' \in [0,\gamma+1]$ and $\widetilde{\rho} = \max\{1, \rho/\varepsilon\}$.

Let $\varepsilon = b^{-1/\alpha}$, then by change of variables we have

$$\|u - u_h\| \le \frac{\varepsilon^\alpha \sin \pi\alpha}{\pi} \int_0^\infty \frac{\rho^\alpha \|w_\rho^\varepsilon - w_{\rho,h}^\varepsilon\|}{\rho^{2\alpha} + 2\rho^\alpha \cos \pi\alpha + 1} d\rho.$$

Theorem 2.2

Suppose Assumption 1.1 holds and $f \in D(\mathcal{A}^{\frac{\delta}{2}})$ with $\delta \in [0,2]$. Denote $\varepsilon = b^{-1/\alpha}$ then we have

$$\|u - u_h\| \le c_\alpha \begin{cases} C(\varepsilon) c_h h^{\min(2\alpha+\delta, 2\gamma)} \|\mathcal{A}^{\frac{\delta}{2}} f\|, & \varepsilon \ge h^2 (b \le h^{-2\alpha}), \\ \varepsilon^\alpha h^{\min(\delta, 2\gamma)} \|\mathcal{A}^{\frac{\delta}{2}} f\|, & \varepsilon < h^2 (b > h^{-2\alpha}), \end{cases}$$

where

$$c_h, C(\varepsilon) = \begin{cases} 1, & \varepsilon^{\alpha + \min(0, \frac{\delta}{2} - \gamma)}, \quad 2\alpha + \delta > 2\gamma, \\ 1 + |\ln h|, & 1, \quad 2\alpha + \delta \le 2\gamma, \end{cases}$$

and c_α is a constant depending on α and is uniformly bounded for $\alpha \in (0,1)$.

Remark 1

For $\gamma = 1$*, Theorem 2.2 can be simplified into the following form*

$$\|u - u_h\| \leq c_\alpha \begin{cases} C'(\varepsilon) c_h h^{\min(2\alpha+\delta,2)} \|\mathcal{A}^{\frac{\delta}{2}} f\|, & b \leq h^{-2\alpha}, \\ b^{-1} h^{\delta} \|\mathcal{A}^{\frac{\delta}{2}} f\|, & b > h^{-2\alpha}, \end{cases}$$

where $C'(\varepsilon) = \varepsilon^{\max(\alpha+\frac{\delta}{2}-1,0)}$ *and* $c_h = 1 + |\ln h|$ *if* $2\alpha+\delta \leq 2$*, otherwise* $c_h = 1$*.*

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Appealing to (2.2), we transplant the domain from $(0, +\infty)$ to $(-\infty, +\infty)$ by letting $\rho^\alpha = e^s$ then

$$u_h = \frac{\sin \pi\alpha}{\alpha\pi} \int_{-\infty}^{\infty} \frac{(1 + e^{-s/\alpha}\mathcal{A}_h)^{-1} f_h}{e^s + 2b\cos\pi\alpha + b^2 e^{-s}} ds. \tag{3.1}$$

To evaluate the integral we apply trapezoidal rule, say

$$u_h \approx u_h^\tau = \frac{\sin \pi\alpha}{\alpha\pi}\tau \sum_{n=-\infty}^{\infty} \frac{(1 + e^{-n\tau/\alpha}\mathcal{A}_h)^{-1} f_h}{e^{n\tau} + 2b\cos\pi\alpha + b^2 e^{-n\tau}}. \tag{3.2}$$

The exponentially decaying property of the integrand allows us to cut the summation to $n = -M, -M+1, \cdots, N-1, N$ with desired accuracy. That is, in practice our scheme is

$$U_{h,\tau}^{M,N} = \frac{\sin \pi\alpha}{\alpha\pi}\tau \sum_{n=-M}^{N} \frac{(1 + e^{-n\tau/\alpha}\mathcal{A}_h)^{-1} f_h}{e^{n\tau} + 2b\cos\pi\alpha + b^2 e^{-n\tau}}. \tag{3.3}$$

Theorem 3.1 (quadrature error)

Let $A(\cdot,\cdot)$ denote the sesquilinear form given in (1.2) which satisfies (1.4) and (1.5), and denote $\mathcal{A}$ the regular accretive operator associate with it and $\mathcal{A}_h$ the corresponding discrete counterpart. Let $\kappa_1 = \alpha(\pi - \arctan\beta)$, $\kappa_2 = (1-\alpha)\pi$, then for $\tau < \pi - \arctan\beta$ we have

$$\|u_h - \bar{u}_h\| \leq \frac{C(\alpha)}{b} C(\tau) e^{-2\pi \min(\kappa_1,\kappa_2)/\tau} \|f_h\|,$$

where

$$C(\tau) = \begin{cases} \sin^{-1}\left(\min\left\{\frac{\kappa_1-\kappa_2}{\alpha} + \frac{1-\alpha}{\alpha}\tau, \frac{\pi}{2}\right\}\right)(1 + |\ln\tau|), & \kappa_1 > \kappa_2, \\ \sin^{-1}\left(\min\left\{\tau, \frac{\pi}{2}\right\}\right)(1 + |\ln(\kappa_2 - \kappa_1 + \alpha\tau)|), & \kappa_1 \leq \kappa_2, \end{cases}$$

and $C(\alpha)$ is a constant depends only on α.

Remark 2

One can observe that in the worst scenario, $C(\tau) = \mathcal{O}(\tau^{-1}(1 + |\ln \tau|))$ *as* $\tau \to 0$, *which is negligible compared with the exponentially decaying term* $e^{-2\pi \min(\kappa_1,\kappa_2)/\tau}$. *In fact, for* α *close to* 1, $C(\tau) = \mathcal{O}(1 + |\ln \tau|)$.

Recalling Theorem 3.1 and using triangle inequality it follows

$$\|u_h - U_{h,\tau}^{M,N}\| \leq \left(\frac{C(\alpha)}{b} C(\tau) e^{-\frac{2\pi \min(\kappa_1,\kappa_2)}{\tau}} + ce^{-N\tau} + c'b^{-2}e^{-(1+\alpha^{-1})M\tau} \right) \|f_h\|.$$

In practice the optimal way to choose τ, M and N are to balance the three terms. Thus, ignoring the negligible coefficients $C(\alpha)$, $C(\tau)$ and c, c' we can choose

$$M = \max\left\{ \frac{\alpha}{\alpha+1} \left(\frac{2\pi \min(\kappa_1,\kappa_2)}{\tau^2} - \frac{\ln b}{\tau} \right), 0 \right\}, \quad N = \frac{2\pi \min(\kappa_1,\kappa_2)}{\tau^2} + \frac{\ln b}{\tau},$$

then for $M > 0$, in terms of number of solves we arrive at

$$\|u_h - U_{h,\tau}^{M,N}\| \leq \frac{C'(\alpha)C(\tau)}{b} e^{-\sqrt{\pi \min(\kappa_1,\kappa_2)\left[(1+\alpha^{-1})M+N\right]}}. \tag{3.4}$$

That is, the error decays root-exponentially with respect to the number of solves.

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Numerical examples

$\mathcal{A}_1$: Laplace operator

$$C(x) = 1, \quad a(x) = 0, \quad r = 0.$$

$\mathcal{A}_2$: General real elliptic operator $r = 0$,

$$C(x) = \begin{pmatrix} 1 + 0.5 \sin \pi x & 0.5 \cos \pi x, \\ 0.5 \sin \pi y & 1 + 0.5 \cos \pi y \end{pmatrix}, \; a(x) = (0.5 + y, 0.5 + x)^T$$

$\mathcal{A}_3$: General complex elliptic operator

$$C(x) = \begin{pmatrix} 0.5 + 5x\mathrm{i} + y & x - y \\ -xy\mathrm{i} & 0.5 + x + 5y\mathrm{i} \end{pmatrix}, \quad a(x) = 0, \quad r = 0.$$

For each operator, we test the convergence under the following three source terms:

A . $\widetilde{H}^2$ source term $f(x,y) = xy(1-x)(1-y) := f_1$;

B . $\widetilde{H}^1$ source term $f(x,y) = (xy)^{0.51}\big((1-x)(1-y)\big)^{0.51} := f_2$;

C . $\widetilde{H}^{0.5-\epsilon}$ source term $f(x,y) = 1 := f_3$.

Spacial error

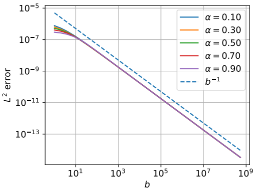
Table 1: Spatial errors with respect to h for $\mathcal{A}_2$

		$\alpha = 0.10$		$\alpha = 0.30$		$\alpha = 0.50$		$\alpha = 0.70$		$\alpha = 0.90$	
	N	E_{L^2}	conv.	E_{L^2}	conv.	E_{L^2}	conv.	E_{L^2}	conv.	E_{L^2}	conv.
f_1	8	2.46e-04	-	2.10e-04	-	1.78e-04	-	1.37e-04	-	9.61e-05	-
	16	5.66e-05	2.12	5.13e-05	2.03	4.51e-05	1.98	3.52e-05	1.96	2.50e-05	1.94
	32	1.35e-05	2.07	1.27e-05	2.01	1.13e-05	1.99	8.90e-06	1.99	6.32e-06	1.98
	64	3.29e-06	2.04	3.16e-06	2.01	2.83e-06	2.00	2.23e-06	2.00	1.58e-06	2.00
	128	8.11e-07	2.02	7.84e-07	2.01	7.02e-07	2.01	5.51e-07	2.01	3.92e-07	2.01
	theor. conv.		2.0		2.0		2.0		2.0		2.0
f_2	8	5.58e-03	-	2.38e-03	-	1.19e-03	-	7.31e-04	-	4.77e-04	-
	16	2.43e-03	1.20	7.76e-04	1.62	3.14e-04	1.92	1.87e-04	1.97	1.23e-04	1.95
	32	1.07e-03	1.18	2.53e-04	1.62	8.14e-05	1.95	4.71e-05	1.99	3.11e-05	1.99
	64	4.78e-04	1.17	8.27e-05	1.61	2.09e-05	1.96	1.18e-05	2.00	7.77e-06	2.00
	128	2.12e-04	1.17	2.70e-05	1.62	5.32e-06	1.98	2.91e-06	2.01	1.92e-06	2.01
	theor. conv.		1.2		1.6		2.0		2.0		2.0
f_3	8	1.21e-01	-	4.15e-02	-	1.43e-02	-	5.92e-03	-	3.03e-03	-
	16	7.78e-02	0.63	2.03e-02	1.03	5.19e-03	1.46	1.70e-03	1.80	7.95e-04	1.93
	32	4.98e-02	0.64	9.77e-03	1.06	1.84e-03	1.49	4.71e-04	1.85	2.02e-04	1.97
	64	3.15e-02	0.66	4.64e-03	1.07	6.49e-04	1.51	1.29e-04	1.87	5.09e-05	1.99
	128	1.96e-02	0.68	2.17e-03	1.09	2.27e-04	1.52	3.46e-05	1.89	1.26e-05	2.01
	theor. conv.		0.7		1.1		1.5		1.9		2.0

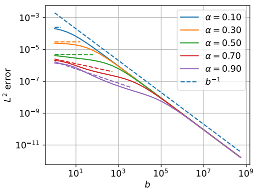
Table 2: Spatial errors with respect to h for $\mathcal{A}_3$

	N	$\alpha = 0.10$ E_{L^2}	conv.	$\alpha = 0.30$ E_{L^2}	conv.	$\alpha = 0.50$ E_{L^2}	conv.	$\alpha = 0.70$ E_{L^2}	conv.	$\alpha = 0.90$ E_{L^2}	conv.
f_1	8	2.29e-04	-	1.71e-04	-	1.28e-04	-	8.50e-05	-	5.10e-05	-
	16	5.23e-05	2.13	4.09e-05	2.07	3.13e-05	2.03	2.10e-05	2.02	1.27e-05	2.01
	32	1.25e-05	2.07	1.01e-05	2.02	7.76e-06	2.01	5.22e-06	2.01	3.16e-06	2.00
	64	3.04e-06	2.03	2.50e-06	2.01	1.93e-06	2.01	1.30e-06	2.00	7.88e-07	2.00
	128	7.52e-07	2.02	6.20e-07	2.01	4.79e-07	2.01	3.22e-07	2.01	1.95e-07	2.01
	theor. conv.		2.0		2.0		2.0		2.0		2.0
f_2	8	5.26e-03	-	2.00e-03	-	9.24e-04	-	5.07e-04	-	2.81e-04	-
	16	2.28e-03	1.21	6.30e-04	1.66	2.34e-04	1.98	1.24e-04	2.03	6.91e-05	2.03
	32	1.01e-03	1.18	2.03e-04	1.64	5.94e-05	1.98	3.06e-05	2.02	1.72e-05	2.01
	64	4.48e-04	1.17	6.60e-05	1.62	1.51e-05	1.97	7.61e-06	2.01	4.28e-06	2.01
	128	1.98e-04	1.17	2.16e-05	1.61	3.84e-06	1.98	1.88e-06	2.02	1.06e-06	2.01
	theor. conv.		1.2		1.6		2.0		2.0		2.0
f_3	8	1.14e-01	-	3.47e-02	-	1.13e-02	-	4.48e-03	-	2.07e-03	-
	16	7.34e-02	0.64	1.68e-02	1.05	3.97e-03	1.51	1.25e-03	1.84	5.27e-04	1.97
	32	4.69e-02	0.65	8.00e-03	1.07	1.38e-03	1.52	3.41e-04	1.87	1.33e-04	1.99
	64	2.96e-02	0.66	3.79e-03	1.08	4.84e-04	1.52	9.24e-05	1.89	3.34e-05	1.99
	128	1.84e-02	0.69	1.77e-03	1.10	1.69e-04	1.52	2.48e-05	1.90	8.29e-06	2.01
	theor. conv.		0.7		1.1		1.5		1.9		2.0

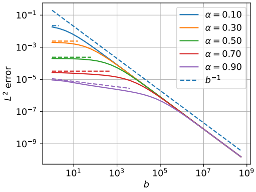
Error with respect to b



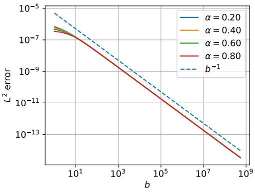
(a) $f = f_1$



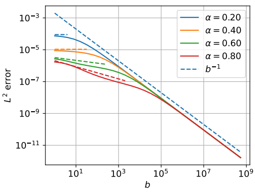
(b) $f = f_2$



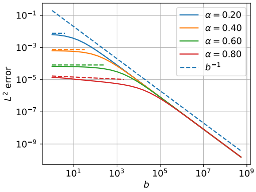
(c) $f = f_3$



(d) $f = f_1$



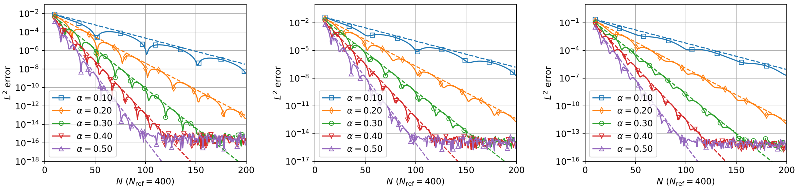
(e) $f = f_2$



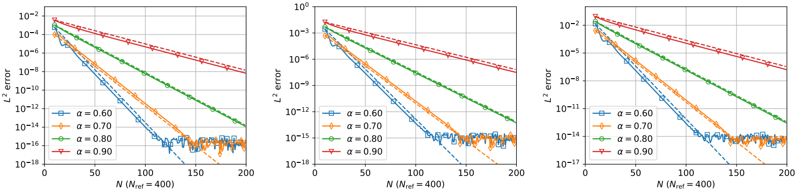
(f) $f = f_3$

Figure 2: The solid lines are spatial errors with respect to b under different f. The dashed lines are theoretical predictions up to a constant multiplier.

Quadrature error

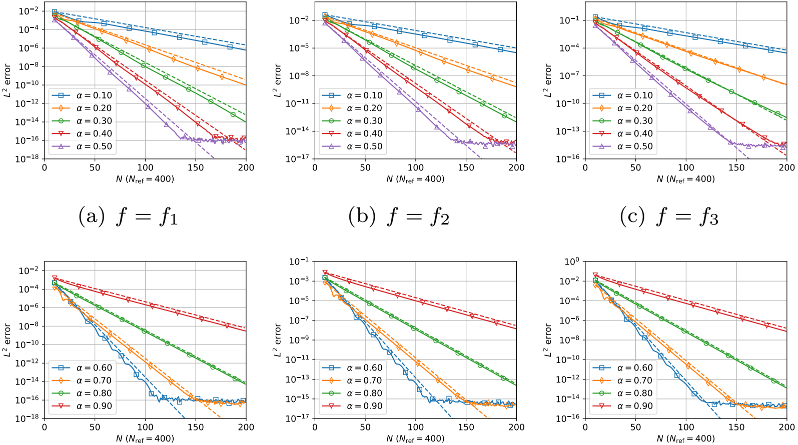


(a) $f = f_1$ (b) $f = f_2$ (c) $f = f_3$



(d) $f = f_1$ (e) $f = f_2$ (f) $f = f_3$

Figure 3: Quadrature errors for $\mathcal{A}_1$ under different f. The dashed lines are $\mathcal{O}(e^{-\frac{2\pi \min(\kappa_1,\kappa_2)}{\tau}})$ with $\kappa_1 = \alpha\pi$, $\kappa_2 = (1-\alpha)\pi$.



(a) $f = f_1$ (b) $f = f_2$ (c) $f = f_3$

(d) $f = f_1$ (e) $f = f_2$ (f) $f = f_3$

Figure 4: Quadrature errors for $\mathcal{A}_3$ under different f. The dashed lines are $\mathcal{O}(e^{-\frac{2\pi \min(\kappa_1,\kappa_2)}{\tau}})$ with $\kappa_1 \approx \alpha(\pi - 1.059)$, $\kappa_2 = (1-\alpha)\pi$.

Thank you for your attention!