Forword

The solutions of some problems are exponentially asymptotical ($\sim e^{\pm t}$), while the solutions of some problems are algebraically asymptotical ($\sim t^{\pm\gamma}$).

Does there exist that the solutions of some problems are logarithmically asymptotical ($\sim (\log t)^{\pm\gamma}$)?

The answer is positive!

Hadamard-type fractional differential equations!

Forword

From the physical phenomena observed and references available, Hadamard fractional calculus is suitable for describing logarithmic asymptotics, e.g., Lomnitz logarithmic creep law of viscoelastic materials [Lomnitz, 1956], ultra slow process [Denisov & Kantz, 2010], life evolution of Populus euphratica, etc.

Discrete formulae for Caputo-Hadamard fractional derivatives and their applications in large time integration

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Outline

Caputo-Hadamard fractional derivative

Definition

Hadamard fractional integral of a given function $f(t)$ with order $\alpha > 0$ is defined by

$$ {}_{H}\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left(\log \frac{t}{\tau} \right)^{\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\tau}, \; t > a > 0. \quad (1.1) $$

The condition $f(t) \in L^1(a,b)$ is presumed. Omitting does not mean no.

Caputo-Hadamard fractional derivative

Definition

Hadamard fractional derivative of a given function $f(t)$ with order $\alpha \, (n-1<\alpha<n\in\mathbb{Z}^+)$ is defined by

$$
\begin{aligned}
{}_H\mathrm{D}_{a,t}^{\alpha} f(t) &= \delta^n \left[{}_H\mathrm{D}_{a,t}^{-(n-\alpha)} f(t) \right] \\
&= \frac{1}{\Gamma(n-\alpha)} \delta^n \int_a^t \left(\log \frac{t}{\tau} \right)^{n-\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\tau}, \; t>a>0,
\end{aligned} \tag{1.2}
$$

where $\delta = t\frac{\mathrm{d}}{\mathrm{d}t}$, $\delta^n = \delta(\delta^{n-1})$, $\delta^0 = I$.

The condition $f(t) \in AC_\delta^n[a,b]$ is presumed. Omitting does not mean no.

Caputo-Hadamard fractional derivative

Definition

Caputo-Hadamard fractional derivative of a given function $f(t)$ with order $\alpha\,(n-1<\alpha<n\in\mathbb{Z}^+)$ and $t>a>0$ is defined by

$$ {}_{CH}\mathrm{D}_{a,t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^n f(s)\frac{\mathrm{d}s}{s}, \qquad (1.3)$$

where $\delta^n f(s)=\left(s\frac{\mathrm{d}}{\mathrm{d}s}\right)^n f(s)=\delta(\delta^{n-1}f(s))$, $\delta^0 f(s)=f(s)$.

The condition $f(t)\in AC_\delta^n[a,b]$ is presumed. Omitting does not mean no. These two kinds of derivatives have following relation,

$$ {}_{CH}\mathrm{D}_{a,t}^{\alpha}f(t)={}_{H}\mathrm{D}_{a,t}^{\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{\delta^k f(a)}{k!}\left(\log\frac{t}{a}\right)^k\right],$$

provided that $\delta^k f(a)=\delta^k f(t)|_{t=a}$ exist for $k=\overline{0,n-1}$.

Caputo-Hadamard fractional derivative

- In 2020, L, Li and Wang got analytical solution to a certain linear fractional partial differential equation with the Caputo-Hadamard derivative by a modified Laplace transform.
- In 2021, L and Li discussed stability and logarithmic decay of the solution of Hadamard-type fractional ordinary differential equation.
- In 2021, L and Li studied the blow-up and global existence of solution to Caputo-Hadamard fractional evolution equation with fractional Laplacian.

Caputo-Hadamard fractional derivative

However, there are few researches on discrete approximation of the Caputo-Hadamard derivative, except that Gohar, L and Li, 2020, and L, Li and Wang, 2020 derived several numerical approximation formulas of the Caputo-Hadamard derivative with $\alpha \in (0, 1)$.

This report comes from Fan, L and Li, 2022.

Two types of subdivision

The partition of the interval $[a, T]$:

$$a = t_0 < t_1 < \cdots < t_N = T.$$

Case A : Uniform partition

$$\begin{aligned} t_k &= t_0 + k\tau, \\ \tau &= t_k - t_{k-1} = \frac{T-a}{N} \ (1 \le k \le N) \, . \end{aligned} \tag{1.4}$$

Case B : Special non-uniform partition (uniform partition in the logarithmic sense)

$$\begin{aligned} t_k &= \exp\left(\log t_0 + k\widetilde{\tau}\right), \ (\text{different nodes}) \\ \widetilde{\tau} &= \log t_k - \log t_{k-1} = \frac{\log T - \log a}{N} \ (1 \le k \le N) \, . \end{aligned} \tag{1.5}$$

Two types of subdivision

For convenience, we define

$$f^k = f(t_k)$$

for the function $f(t)$ on $[a, T]$ and introduce the following operator

$$\nabla_{\log,t} f^{k-\frac{1}{2}} = \frac{f^k - f^{k-1}}{\log \frac{t_k}{t_{k-1}}}.$$

Uniform partition

The partition of the interval $[a, T]$:

$$a = t_0 < t_1 < \cdots < t_N = T.$$

Case A : Uniform partition

$$\begin{aligned} t_k &= t_0 + k\tau, \\ \tau &= t_k - t_{k-1} = \frac{T-a}{N} \; (1 \le k \le N)\,. \end{aligned} \tag{2.1}$$

L1-2 formula with order $0 < \alpha < 1$

We denote the linear interpolation function of $f(t)$ as $L_{\log,1,j}f(t)$ on $[t_{j-1}, t_j] (1 \le j \le N)$ by $(t_{j-1}, f(t_{j-1}))$, $(t_j, f(t_j))$, that is,

$$L_{\log,1,j}f(t) = \frac{\log \frac{t}{t_j}}{\log \frac{t_{j-1}}{t_j}} f^{j-1} + \frac{\log \frac{t}{t_{j-1}}}{\log \frac{t_j}{t_{j-1}}} f^j, \tag{2.2}$$

and the truncation error on $[t_{j-1}, t_j]$ is

$$r_1^j(t) = f(t) - L_{\log,1,j}f(t) = \frac{1}{2}\delta^2 f(\eta_j) \log \frac{t}{t_{j-1}} \log \frac{t}{t_j}, \tag{2.3}$$

where $\eta_j \in (t_{j-1}, t_j)$.

L1-2 formula with order $0 < \alpha < 1$

We obtain quadratic interpolation function $L_{\log,2,j}f(t)$ on $[t_{j-1}, t_j]$ $(2 \le j \le N)$ using $(t_{j-2}, f(t_{j-2}))$, $(t_{j-1}, f(t_{j-1}))$, $(t_j, f(t_j))$,

$$\begin{aligned}
&L_{\log,2,j}f(t) \\
=&\frac{\log\frac{t}{t_{j-1}}\log\frac{t}{t_j}}{\log\frac{t_{j-2}}{t_{j-1}}\log\frac{t_{j-2}}{t_j}}f^{j-2} + \frac{\log\frac{t}{t_{j-2}}\log\frac{t}{t_j}}{\log\frac{t_{j-1}}{t_{j-2}}\log\frac{t_{j-1}}{t_j}}f^{j-1} + \frac{\log\frac{t}{t_{j-2}}\log\frac{t}{t_{j-1}}}{\log\frac{t_j}{t_{j-2}}\log\frac{t_j}{t_{j-1}}}f^j \\
=&L_{\log,1,j}f(t) + \frac{\nabla_{\log,t}f^{j-\frac{1}{2}} - \nabla_{\log,t}f^{j-\frac{3}{2}}}{\log\frac{t_j}{t_{j-2}}}\log\frac{t}{t_{j-1}}\log\frac{t}{t_j}.
\end{aligned} \tag{2.4}$$

The truncation error on $[t_{j-1}, t_j]$ is as follows

$$\begin{aligned}
r_2^j(t) =& f(t) - L_{\log,2,j}f(t) \\
=&\frac{1}{6}\delta^3 f(\xi_j)\log\frac{t}{t_{j-2}}\log\frac{t}{t_{j-1}}\log\frac{t}{t_j}, \quad \xi_j \in (t_{j-2}, t_j).
\end{aligned} \tag{2.5}$$

L1-2 formula with order $0 < \alpha < 1$

From (2.2)-(2.5), we can arrive at

$$\begin{aligned} &\delta\left(L_{\log,2,j} f(t)\right) \\ &= \delta\left(L_{\log,1,j} f(t)\right) + \frac{\nabla_{\log,t} f^{j-\frac{1}{2}} - \nabla_{\log,t} f^{j-\frac{3}{2}}}{\log \frac{t_j}{t_{j-2}}} \log \frac{t^2}{t_j t_{j-1}}, \end{aligned} \tag{2.6}$$

$$\delta\left(L_{\log,1,j} f(t)\right) = \nabla_{\log,t} f^{j-\frac{1}{2}}.$$

L1-2 formula with order $0 < \alpha < 1$

$$
\begin{aligned}
&{}_{CH}\mathrm{D}^{\alpha}_{a,t} f(t)\Big|_{t=t_k} \\
&= \frac{1}{\Gamma(1-\alpha)} \left\{ \int_{t_0}^{t_1} \left(\log \frac{t_k}{s}\right)^{-\alpha} \delta\left(L_{\log,1,1} f(s)\right) \frac{\mathrm{d}s}{s} \right. \\
&\qquad \left. + \sum_{j=2}^{k} \int_{t_{j-1}}^{t_j} \left(\log \frac{t_k}{s}\right)^{-\alpha} \delta\left(L_{\log,2,j} f(s)\right) \frac{\mathrm{d}s}{s} \right\} \\
&\quad + \frac{1}{\Gamma(1-\alpha)} \left\{ \int_{t_0}^{t_1} \left(\log \frac{t_k}{s}\right)^{-\alpha} \delta\left(r_1^1(s)\right) \frac{\mathrm{d}s}{s} \right. \\
&\qquad \left. + \sum_{j=2}^{k} \int_{t_{j-1}}^{t_j} \left(\log \frac{t_k}{s}\right)^{-\alpha} \delta\left(r_2^j(s)\right) \frac{\mathrm{d}s}{s} \right\} \\
&= {}_{CH}\mathbb{D}^{\alpha}_{a,t} f^k + R^k.
\end{aligned}
\tag{2.7}
$$

L1-2 formula with order $0 < \alpha < 1$

So L1-2 formula of Caputo-Hadamard fractional derivative with $\alpha \in (0,1)$ is obtained as follows:

$$
\begin{aligned}
&{}_{CH}\mathbb{D}^{\alpha}_{a,t} f^k \\
&= {}_{CH}\mathrm{D}^{\alpha}_{a,t} f^k - \frac{1}{\Gamma(2-\alpha)} \sum_{j=2}^{k} b^{(\alpha)}_{j,k} \left(\nabla_{\log,t} f^{j-\frac{1}{2}} - \nabla_{\log,t} f^{j-\frac{3}{2}} \right) \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^{k} c^{(\alpha)}_{j,k} \left(f^j - f^{j-1} \right),
\end{aligned}
\tag{2.8}
$$

where

$$
{}_{CH}\mathrm{D}^{\alpha}_{a,t} f^k = \frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^{k} a^{(\alpha)}_{j,k} \nabla_{\log,t} f^{j-\frac{1}{2}} \ (\text{L1 formula}),
$$

L1-2 formula with order $0 < \alpha < 1$

and

$$
c_{j,k}^{(\alpha)} = \begin{cases} \frac{1}{\log\frac{t_1}{t_0}}(a_{1,k}^{(\alpha)} + b_{2,k}^{(\alpha)}), \ j = 1, \\ \frac{1}{\log\frac{t_j}{t_{j-1}}}(a_{j,k}^{(\alpha)} - b_{j,k}^{(\alpha)} + b_{j+1,k}^{(\alpha)}), \ 2 \le j \le k-1, \\ \frac{1}{\log\frac{t_k}{t_{k-1}}}(a_{k,k}^{(\alpha)} - b_{k,k}^{(\alpha)}), \ j = k, \end{cases}
$$

$$
a_{j,k}^{(\alpha)} = \left(\log\frac{t_k}{t_{j-1}}\right)^{1-\alpha} - \left(\log\frac{t_k}{t_j}\right)^{1-\alpha}, \tag{2.9}
$$

$$
\begin{aligned} b_{j,k}^{(\alpha)} = \Bigg\{ & \log\frac{t_j}{t_{j-1}}\left[\left(\log\frac{t_k}{t_j}\right)^{1-\alpha} + \left(\log\frac{t_k}{t_{j-1}}\right)^{1-\alpha}\right] \\ & + \frac{2}{2-\alpha}\left[\left(\log\frac{t_k}{t_j}\right)^{2-\alpha} - \left(\log\frac{t_k}{t_{j-1}}\right)^{2-\alpha}\right]\Bigg\}\frac{1}{\log\frac{t_j}{t_{j-2}}}. \end{aligned}
$$

L1-2 formula with order $0 < \alpha < 1$

Theorem

Assuming $f(t) \in C^3[a,T]$ and $0 < \alpha < 1$, for uniform partition of the interval $[a,T]$ with $\tau = t_k - t_{k-1}$, the truncation errors R^k $(1 \le k \le N)$ in (2.7) satisfy

$$
\begin{aligned}
|R^1| &\le \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} |\delta^2 f(t)| \Big(\log\frac{t_1}{t_0}\Big)^{2-\alpha}, \quad k = 1, \\
|R^k| &\le \frac{\alpha}{8\Gamma(1-\alpha)} \max_{t_0 \le t \le t_1} |\delta^2 f(t)| \Big(\log\frac{t_k}{t_1}\Big)^{-1-\alpha} \Big(\log\frac{t_1}{t_0}\Big)^{3} \\
&\quad + \frac{1}{12\Gamma(1-\alpha)} \max_{t_0 \le t \le t_{k-1}} |\delta^3 f(t)| \max_{1 \le l \le k-1} \Big(\log\frac{t_l}{t_{l-1}}\Big)^{3} \Big(\log\frac{t_k}{t_{k-1}}\Big)^{-\alpha} \\
&\quad + \frac{\alpha(5-\alpha)}{6\Gamma(4-\alpha)} \max_{t_{k-2} \le t \le t_k} |\delta^3 f(t)| \max_{k-1 \le l \le k} \Big(\log\frac{t_l}{t_{l-1}}\Big)^{3-\alpha}, \quad k \ge 2.
\end{aligned}
\tag{2.10}
$$

L1-2 formula with order $0 < \alpha < 1$

Lemma

For $\alpha \in (0,1)$, coefficients $b_{j,k}^{(\alpha)}$ $(2 \le j \le k,\ 2 \le k \le N)$ in (2.9) with $t_j = t_0 + j\tau\,(0 \le j \le k)$ are negative.

Lemma

The inequalities with $t_j = t_0 + j\tau\,(1 \le j \le k-2,\ 3 \le k \le N)$ hold

$$\log \frac{t_{j+2}}{t_{j+1}} \log \frac{t_j}{t_{j-1}} - \left(\log \frac{t_{j+1}}{t_j}\right)^2 > 0. \tag{2.11}$$

L1-2 formula with order $0 < \alpha < 1$

Lemma

For any $\alpha \in (0,1)$ and $t_j = t_0 + j\tau\,(0 \le j \le k)$, the inequalities with $a_{j,k}^{(\alpha)}$ and $b_{j,k}^{(\alpha)}$ in (2.9) hold

$$a_{j,k}^{(\alpha)} + b_{j+1,k}^{(\alpha)} > 0, \quad 1 \le j \le k-2,\ 3 \le k \le N. \tag{2.12}$$

Theorem

For any $\alpha \in (0,1)$ and $t_j = t_0 + j\tau\,(0 \le j \le k)$, coefficients $c_{j,k}^{(\alpha)}\,(1 \le j \le k,\ 1 \le k \le N)$ in (2.9) satisfy

$$c_{j,k}^{(\alpha)} > 0, \quad j \ne k-1. \tag{2.13}$$

L1-2 formula with order $0 < \alpha < 1$

Remark

For $j = k-1$, $a = 1$, $T = 2$ and $N = 25$, we find that the sign of $c^{(\alpha)}_{k-1,k}$ $(3 \le k \le N)$ with $\alpha = 0.1, 0.685, 0.686, 0.9$ can change.

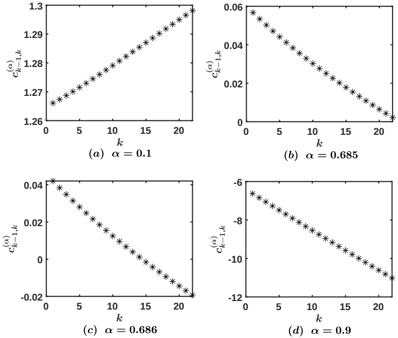

Figure: The values of $c^{(\alpha)}_{k-1,k}$ with the uniform partition.

L1-2 formula with order $0 < \alpha < 1$

Theorem

For order $\alpha \in (0,1)$ and sufficiently small step τ, the coefficients $c_{j,k}^{(\alpha)}\,(1 \le j \le k,\ 2 \le k \le N)$ in (2.9) with $t_j = t_0 + j\tau\,(0 \le j \le k)$ satisfy

$(1)\ c_{k,k}^{(\alpha)} > \left|c_{k-1,k}^{(\alpha)}\right|\,(k \ge 2)\,,$

$(2)\ c_{k,k}^{(\alpha)} > c_{k-2,k}^{(\alpha)}\,(k \ge 3)\,,$

$(3)\ c_{k-2,k}^{(\alpha)} > c_{k-3,k}^{(\alpha)} > \cdots > c_{1,k}^{(\alpha)}\,(k \ge 4).$

L2-1_σ formula with $0 < \alpha < 1$

We denote the quadratic interpolation function $\Pi_{\log,2,j} f(t)$ of $f(t)$ in the sense of logarithm on $[t_{j-1}, t_j]$ $(1 \le j \le k, 1 \le k \le N-1)$ by using the points $(t_{j-1}, f(t_{j-1}))$, $(t_j, f(t_j))$, $(t_{j+1}, f(t_{j+1}))$,

$$\begin{aligned}\Pi_{\log,2,j} f(t) = &\frac{\log\frac{t}{t_j}\log\frac{t}{t_{j+1}}}{\log\frac{t_{j-1}}{t_j}\log\frac{t_{j-1}}{t_{j+1}}} f^{j-1} + \frac{\log\frac{t}{t_{j-1}}\log\frac{t}{t_{j+1}}}{\log\frac{t_j}{t_{j-1}}\log\frac{t_j}{t_{j+1}}} f^{j} \\ &+ \frac{\log\frac{t}{t_{j-1}}\log\frac{t}{t_j}}{\log\frac{t_{j+1}}{t_{j-1}}\log\frac{t_{j+1}}{t_j}} f^{j+1},\end{aligned} \tag{2.14}$$

and the truncation error on $[t_{j-1}, t_j]$,

$$\begin{aligned} r_2^j(t) =& f(t) - \Pi_{\log,2,j} f(t) \\ =& \frac{1}{6}\delta^3 f(\xi_j) \log\frac{t}{t_{j-1}} \log\frac{t}{t_j} \log\frac{t}{t_{j+1}}, \end{aligned} \tag{2.15}$$

where $\xi_j \in (t_{j-1}, t_{j+1})$.

L2-1$_\sigma$ formula with $0 < \alpha < 1$

Let $\sigma = 1 - \frac{\alpha}{2}$ be a fixed constant and $t_{k+\sigma} = t_k + \sigma\tau$. Then we take $\Pi_{\log,1,k+1} f(t)$ as the linear interpolation function of $f(t)$ on the interval $[t_k, t_{k+\sigma}]$ $(k = 0, 1, \cdots, N-1)$ in the logarithmic sense, using the points $(t_k, f(t_k))$, $(t_{k+1}, f(t_{k+1}))$ to get

$$\Pi_{\log,1,k+1} f(t) = \frac{\log \frac{t}{t_{k+1}}}{\log \frac{t_k}{t_{k+1}}} f^k + \frac{\log \frac{t}{t_k}}{\log \frac{t_{k+1}}{t_k}} f^{k+1}, \tag{2.16}$$

and the truncation error on $[t_k, t_{k+\sigma}]$,

$$\begin{aligned} r_1^{k+1}(t) =& f(t) - \Pi_{\log,1,k+1} f(t) \\ =& \frac{1}{2}\delta^2 f(\eta_{k+1}) \log \frac{t}{t_k} \log \frac{t}{t_{k+1}}, \;\; \eta_{k+1} \in (t_k, t_{k+1}). \end{aligned} \tag{2.17}$$

L2-1$_\sigma$ formula with $0 < \alpha < 1$

Thus, we can arrive at

$$\begin{aligned}
&\delta\left(\Pi_{\log,2,j} f(t)\right) \\
&= \nabla_{\log,t} f^{j-\frac{1}{2}} + \frac{\nabla_{\log,t} f^{j+\frac{1}{2}} - \nabla_{\log,t} f^{j-\frac{1}{2}}}{\log \frac{t_{j+1}}{t_{j-1}}} \log \frac{t^2}{t_j t_{j-1}},
\end{aligned} \tag{2.18}$$

$$\delta\left(\Pi_{\log,1,k+1} f(t)\right) = \nabla_{\log,t} f^{k+\frac{1}{2}}.$$

L2-1$_\sigma$ formula with $0 < \alpha < 1$

$$
\begin{aligned}
& {}_{CH}\mathrm{D}^{\alpha}_{a,t} f(t)\Big|_{t=t_{k+\sigma}} \\
&= \frac{1}{\Gamma(1-\alpha)} \left\{ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left(\log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \delta \left(\Pi_{\log,2,j} f(s) \right) \frac{\mathrm{d}s}{s} \right. \\
&\quad \left. + \int_{t_k}^{t_{k+\sigma}} \left(\log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \delta \left(\Pi_{\log,1,k+1} f(s) \right) \frac{\mathrm{d}s}{s} \right\} \\
&\quad + \frac{1}{\Gamma(1-\alpha)} \left\{ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left(\log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \delta \left(r_2^j(s) \right) \frac{\mathrm{d}s}{s} \right. \\
&\quad \left. + \int_{t_k}^{t_{k+\sigma}} \left(\log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \delta \left(r_1^{k+1}(s) \right) \frac{\mathrm{d}s}{s} \right\} \\
&= {}_{CH}\mathfrak{D}^{\alpha}_{a,t} f^{k+\sigma} + R^{k+\sigma}.
\end{aligned}
\tag{2.19}
$$

L2-1$_\sigma$ formula with $0 < \alpha < 1$

By means of (2.18), we can obtain L2-1$_\sigma$ formula with $\alpha \in (0,1)$

$$
\begin{aligned}
&{}_{CH}\mathfrak{D}^{\alpha}_{a,t} f^{k+\sigma} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \Bigg\{ \nabla_{\log,t} f^{j-\frac{1}{2}} \int_{t_{j-1}}^{t_j} \left(\log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \frac{\mathrm{d}s}{s} \\
&\quad + \frac{\nabla_{\log,t} f^{j+\frac{1}{2}} - \nabla_{\log,t} f^{j-\frac{1}{2}}}{\log \frac{t_{j+1}}{t_{j-1}}} \int_{t_{j-1}}^{t_j} \left(\log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \log \frac{s^2}{t_{j-1} t_j} \frac{\mathrm{d}s}{s} \Bigg\} \\
&\quad + \frac{\nabla_{\log,t} f^{k+\frac{1}{2}}}{\Gamma(1-\alpha)} \int_{t_k}^{t_{k+\sigma}} \left(\log \frac{t_{k+\sigma}}{s} \right)^{-\alpha} \frac{\mathrm{d}s}{s} \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^{k+1} c_{j,k}^{(\alpha,\sigma)} \left(f^j - f^{j-1} \right),
\end{aligned}
\tag{2.20}
$$

L2-1$_\sigma$ formula with $0 < \alpha < 1$

where

$$c_{j,k}^{(\alpha,\sigma)} = \begin{cases} \frac{1}{\log\frac{t_1}{t_0}}\left(a_{1,k}^{(\alpha,\sigma)} - b_{1,k}^{(\alpha,\sigma)}\right), & j = 1, \\ \frac{1}{\log\frac{t_j}{t_{j-1}}}\left(a_{j,k}^{(\alpha,\sigma)} + b_{j-1,k}^{(\alpha,\sigma)} - b_{j,k}^{(\alpha,\sigma)}\right), & 2 \le j \le k, \\ \frac{1}{\log\frac{t_{k+1}}{t_k}}\left(b_{k,k}^{(\alpha,\sigma)} + \left(\log\frac{t_{k+\sigma}}{t_k}\right)^{1-\alpha}\right), & j = k+1, \end{cases}$$

$$a_{j,k}^{(\alpha,\sigma)} = \left(\log\frac{t_{k+\sigma}}{t_{j-1}}\right)^{1-\alpha} - \left(\log\frac{t_{k+\sigma}}{t_j}\right)^{1-\alpha},$$

$$\begin{aligned} b_{j,k}^{(\alpha,\sigma)} = \Bigg\{ & \frac{2}{2-\alpha}\left[\left(\log\frac{t_{k+\sigma}}{t_{j-1}}\right)^{2-\alpha} - \left(\log\frac{t_{k+\sigma}}{t_j}\right)^{2-\alpha}\right] \\ & - \log\frac{t_j}{t_{j-1}}\left[\left(\log\frac{t_{k+\sigma}}{t_j}\right)^{1-\alpha} + \left(\log\frac{t_{k+\sigma}}{t_{j-1}}\right)^{1-\alpha}\right]\Bigg\} \frac{1}{\log\frac{t_{j+1}}{t_{j-1}}}. \end{aligned} \tag{2.21}$$

L2-1$_\sigma$ formula with $0 < \alpha < 1$

Theorem

Letting $f(t) \in C^3[a,T]$ *and* $\alpha \in (0,1)$, *for the fixed* $\sigma = 1 - \frac{\alpha}{2}$ *and sufficiently small* $\tau = \frac{T-a}{N}$, *the truncation errors* $R^{k+\sigma}$ $(0 \le k \le N-1)$ *in (2.19) with* $t_k = t_0 + k\tau$ *and* $t_{k+\sigma} = t_k + \sigma\tau$ *satisfy*

$$\begin{aligned}\left|R^{k+\sigma}\right| \le & \frac{1}{12\Gamma(1-\alpha)} \max_{t_0 \le t \le t_{k+1}} |\delta^3 f(t)| \max_{1 \le l \le k+1} \Big(\log \frac{t_l}{t_{l-1}}\Big)^3 \Big(\log \frac{t_{k+\sigma}}{t_k}\Big)^{-\alpha} \\ & + \left\{ \frac{1}{\Gamma(3-\alpha)} \left(1 + \frac{\sigma(1-\sigma)}{2}\right) \max_{t_k \le t \le t_{k+1}} |\delta^2 f(t)| \right. \\ & \left. + \frac{1}{6\Gamma(2-\alpha)} \max_{t_k \le t \le t_{k+1}} |\delta^3 f(t)| \right\} \Big(\log \frac{t_{k+1}}{t_k}\Big)^2 \Big(\log \frac{t_{k+\sigma}}{t_k}\Big)^{1-\alpha}. \end{aligned} \tag{2.22}$$

L2-1$_\sigma$ formula with $0 < \alpha < 1$

Lemma

For order $\alpha \in (0,1)$ *and* $t_j = t_0 + j\tau\,(0 \le j \le k+1)$, *coefficients* $b_{j,k}^{(\alpha,\sigma)}$ *defined in (2.21) satisfy*

$$b_{j,k}^{(\alpha,\sigma)} > 0,\ \ 1 \le j \le k,\ 1 \le k \le N-1. \tag{2.23}$$

Lemma

For $\alpha \in (0,1)$ *and* $t_j = t_0 + j\tau\,(0 \le j \le k+1)$, *the inequalities with* $a_{j,k}^{(\alpha,\sigma)}$ *and* $b_{j,k}^{(\alpha,\sigma)}$ *in (2.21) hold*

$$a_{j,k}^{(\alpha,\sigma)} - b_{j,k}^{(\alpha,\sigma)} > 0,\ \ 1 \le j \le k,\ 1 \le k \le N-1. \tag{2.24}$$

L2-1$_\sigma$ formula with $0 < \alpha < 1$

Theorem

For any order $\alpha \in (0,1)$ and $t_j = t_0 + j\tau\,(0 \le j \le k+1)$, coefficients $c_{j,k}^{(\alpha,\sigma)}$ defined in (2.21) satisfy

$$c_{j,k}^{(\alpha,\sigma)} > 0,\ 1 \le j \le k+1,\, 0 \le k \le N-1. \tag{2.25}$$

Theorem

For order $\alpha \in (0,1)$ and sufficiently small τ, coefficients $c_{j,k}^{(\alpha,\sigma)}$ $(1 \le k \le N-1)$ defined in (2.21) satisfy

$$c_{k+1,k}^{(\alpha,\sigma)} > c_{k,k}^{(\alpha,\sigma)} > c_{k-1,k}^{(\alpha,\sigma)} > \cdots > c_{2,k}^{(\alpha,\sigma)} > c_{1,k}^{(\alpha,\sigma)}. \tag{2.26}$$

H2N2 formula with $1 < \alpha < 2$

Let $t_{k-\frac{1}{2}} = \frac{t_{k-1}+t_k}{2}$, i.e., the arithmetic mean of t_{k-1} and t_k. We show the quadratic Hermite interpolation $H_{\log,2,0}f(t)$ of $f(t)$ on the interval $\left[t_0, t_{\frac{1}{2}}\right]$ in the sense of logarithm using the three points $(t_0, f(t_0))$, $(t_1, f(t_1))$, $(t_0, \delta f(t_0))$,

$$
\begin{aligned}
& H_{\log,2,0}f(t) \\
&= f(t_0) + \delta f(t_0) \log\frac{t}{t_0} + \frac{\nabla_{\log,t} f^{\frac{1}{2}} - \delta f(t_0)}{\log\frac{t_1}{t_0}} \left(\log\frac{t}{t_0}\right)^2,
\end{aligned}
\quad (2.27)
$$

and the truncation error on $\left[t_0, t_{\frac{1}{2}}\right]$,

$$
\begin{aligned}
R_H(t) =& f(t) - H_{\log,2,0}f(t) \\
=& \frac{1}{6}\delta^3 f(\xi_0)\left(\log\frac{t}{t_0}\right)^2 \log\frac{t}{t_1}, \quad \xi_0 \in (t_0, t_1).
\end{aligned}
\quad (2.28)
$$

H2N2 formula with $1 < \alpha < 2$

Similarly, on the interval $\left[t_{j-\frac{1}{2}}, t_{j+\frac{1}{2}}\right]$ $(j = 1, 2, \cdots, N-1)$, we obtain quadratic Newton interpolation $N_{\log,2,j}f(t)$ of the function $f(t)$ in the logarithmic sense, by means of the points $(t_{j-1}, f(t_{j-1}))$, $(t_j, f(t_j))$, $(t_{j+1}, f(t_{j+1}))$,

$$\begin{aligned} N_{\log,2,j}f(t) &= f(t_{j-1}) + \nabla_{\log,t} f^{j-\frac{1}{2}} \log\frac{t}{t_{j-1}} \\ &\quad + \frac{\nabla_{\log,t} f^{j+\frac{1}{2}} - \nabla_{\log,t} f^{j-\frac{1}{2}}}{\log\frac{t_{j+1}}{t_{j-1}}} \log\frac{t}{t_{j-1}} \log\frac{t}{t_j}, \end{aligned} \tag{2.29}$$

and the truncation error on $\left[t_{j-\frac{1}{2}}, t_{j+\frac{1}{2}}\right]$ $(1 \le j \le N)$,

$$\begin{aligned} R_N^j(t) &= f(t) - N_{\log,2,j}f(t) \\ &= \frac{1}{6}\delta^3 f(\xi_j) \log\frac{t}{t_{j-1}} \log\frac{t}{t_j} \log\frac{t}{t_{j+1}}, \xi_j \in (t_{j-1}, t_{j+1}). \end{aligned} \tag{2.30}$$

H2N2 formula with $1 < \alpha < 2$

Therefore, we have

$$\begin{cases} \delta^2\left(H_{\log,2,0}f(t)\right) = \dfrac{2\left(\nabla_{\log,t}f^{\frac{1}{2}} - \delta f(t_0)\right)}{\log\frac{t_1}{t_0}}, \\ \delta^2\left(N_{\log,2,j}f(t)\right) = \dfrac{2\left(\nabla_{\log,t}f^{j+\frac{1}{2}} - \nabla_{\log,t}f^{j-\frac{1}{2}}\right)}{\log\frac{t_{j+1}}{t_{j-1}}}. \end{cases} \tag{2.31}$$

H2N2 formula with $1 < \alpha < 2$

$$
\begin{aligned}
&{}_{CH}\mathrm{D}^{\alpha}_{a,t} f(t)\Big|_{t=t_{k-\frac{1}{2}}} \\
&= \frac{1}{\Gamma(2-\alpha)}\left\{ \int_{t_0}^{t_{\frac{1}{2}}} \left(\log\frac{t_{k-\frac{1}{2}}}{s}\right)^{1-\alpha} \delta^2 H_{\log,2,0} f(s) \frac{\mathrm{d}s}{s} \right. \\
&\quad \left. + \sum_{j=1}^{k-1} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \left(\log\frac{t_{k-\frac{1}{2}}}{s}\right)^{1-\alpha} \delta^2 N_{\log,2,j} f(s) \frac{\mathrm{d}s}{s} \right\} \\
&\quad + \frac{1}{\Gamma(2-\alpha)}\left\{ \int_{t_0}^{t_{\frac{1}{2}}} \left(\log\frac{t_{k-\frac{1}{2}}}{s}\right)^{1-\alpha} \delta^2 R_H(s) \frac{\mathrm{d}s}{s} \right. \\
&\quad \left. + \sum_{j=1}^{k-1} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \left(\log\frac{t_{k-\frac{1}{2}}}{s}\right)^{1-\alpha} \delta^2 R_N^j(s) \frac{\mathrm{d}s}{s} \right\} \\
&= {}_{CH}\mathbb{D}^{\alpha}_{a,t} f^{k-\frac{1}{2}} + R^{k-\frac{1}{2}}.
\end{aligned}
\tag{2.32}
$$

H2N2 formula with $1 < \alpha < 2$

By formula (2.31), we can arrive at H2N2 formula with $\alpha \in (1, 2)$

$$
\begin{aligned}
&{}_{CH}\mathbb{D}^{\alpha}_{a,t} f^{k-\frac{1}{2}} \\
&= \frac{1}{\Gamma(2-\alpha)} \frac{2(\nabla_{\log,t} f^{\frac{1}{2}} - \delta f(t_0))}{\log \frac{t_1}{t_0}} \int_{t_0}^{t_{\frac{1}{2}}} \left(\log \frac{t_{k-\frac{1}{2}}}{s} \right)^{1-\alpha} \frac{\mathrm{d}s}{s} \\
&\quad + \frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^{k-1} \frac{2(\nabla_{\log,t} f^{j+\frac{1}{2}} - \nabla_{\log,t} f^{j-\frac{1}{2}})}{\log \frac{t_{j+1}}{t_{j-1}}} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \left(\log \frac{t_{k-\frac{1}{2}}}{s} \right)^{1-\alpha} \frac{\mathrm{d}s}{s} \\
&= \frac{2}{\Gamma(3-\alpha)} \sum_{j=1}^{k} c^{(\alpha)}_{j,k} (f^j - f^{j-1}) - \frac{2}{\Gamma(3-\alpha)} a^{(\alpha)}_{0,k}\, \delta f(t_0),
\end{aligned}
\tag{2.33}
$$

H2N2 formula with $1 < \alpha < 2$

where

$$c_{j,k}^{(\alpha)} = \begin{cases} \dfrac{1}{\log \frac{t_1}{t_0}}\left(a_{0,k}^{(\alpha)} - a_{1,k}^{(\alpha)}\right), & j = 1, \\ \dfrac{1}{\log \frac{t_j}{t_{j-1}}}\left(a_{j-1,k}^{(\alpha)} - a_{j,k}^{(\alpha)}\right), & 2 \le j \le k-1, \\ \dfrac{1}{\log \frac{t_k}{t_{k-1}}} a_{k-1,k}^{(\alpha)}, & j = k, \end{cases}$$

$$a_{j,k}^{(\alpha)} = \begin{cases} \dfrac{\left(\log \frac{t_{k-\frac{1}{2}}}{t_0}\right)^{2-\alpha} - \left(\log \frac{t_{k-\frac{1}{2}}}{t_{\frac{1}{2}}}\right)^{2-\alpha}}{\log \frac{t_1}{t_0}}, & j = 0, \\ \dfrac{\left(\log \frac{t_{k-\frac{1}{2}}}{t_{j-\frac{1}{2}}}\right)^{2-\alpha} - \left(\log \frac{t_{k-\frac{1}{2}}}{t_{j+\frac{1}{2}}}\right)^{2-\alpha}}{\log \frac{t_{j+1}}{t_{j-1}}}, & 1 \le j \le k-1. \end{cases} \tag{2.34}$$

H2N2 formula with $1 < \alpha < 2$

Theorem

Let $f(t) \in C^3[a,T]$ *and* $1 < \alpha < 2$, *for the sufficiently small* τ, *the truncation errors* $R^{k-\frac{1}{2}}$ $(1 \le k \le N)$ *in (2.32) hold*

$$
\begin{aligned}
|R^{k-\frac{1}{2}}| &\le \frac{1}{\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} |\delta^3 f(t)| \log \frac{t_1}{t_0} \Big(\log \frac{t_{\frac{1}{2}}}{t_0} \Big)^{2-\alpha}, \quad k = 1, \\
|R^{k-\frac{1}{2}}| &\le \frac{1}{\Gamma(2-\alpha)} \max_{t_0 \le t \le t_{k-1}} |\delta^3 f(t)| \Big\{ \frac{17}{6} \max_{1 \le l \le k-1} \Big(\log \frac{t_l}{t_{l-1}} \Big)^2 \Big(\log \frac{t_{k-\frac{1}{2}}}{t_{k-\frac{3}{2}}} \Big)^{1-\alpha} \\
&\quad + \frac{3}{8} \log \frac{T}{a} \max_{1 \le l \le k-1} \Big(\log \frac{t_l}{t_{l-1}} \Big)^3 \Big(\log \frac{t_{k-\frac{1}{2}}}{t_{k-\frac{3}{2}}} \Big)^{-\alpha} \Big\} \\
&\quad + \frac{1}{\Gamma(3-\alpha)} \max_{t_{k-2} \le t \le t_k} |\delta^3 f(t)| \max_{1 \le l \le k-1} \log \frac{t_l}{t_{l-1}} \Big(\log \frac{t_{k-\frac{1}{2}}}{t_{k-\frac{3}{2}}} \Big)^{2-\alpha}, k \ge 2.
\end{aligned} \tag{2.35}
$$

H2N2 formula with $1 < \alpha < 2$

Theorem

For $\alpha \in (1,2)$, coefficients $c_{j,k}^{(\alpha)}$ in (2.34) with $t_j = t_0 + j\tau$ and $t_{j-\frac{1}{2}} = t_{j-1} + \frac{1}{2}\tau\,(1 \le j \le k,\ 1 \le k \le N)$ hold

$$c_{k,k}^{(\alpha)} > 0, \quad c_{j,k}^{(\alpha)} < 0\,(2 \le j \le k-1). \tag{2.36}$$

Theorem

For $\alpha \in (1,2)$ and sufficiently small τ, coefficients $c_{j,k}^{(\alpha)}$ in (2.34) with $t_j = t_0 + j\tau$ and $t_{j-\frac{1}{2}} = t_{j-1} + \frac{1}{2}\tau\,(1 \le j \le k,\ 1 \le k \le N)$ hold

(1) $c_{1,k}^{(\alpha)} > c_{2,k}^{(\alpha)} > c_{3,k}^{(\alpha)} > \cdots > c_{k-1,k}^{(\alpha)}$;

(2) $c_{k,k}^{(\alpha)} > \left|c_{k-1,k}^{(\alpha)}\right|\,(k \ge 2)$.

Special non-uniform partition (uniform partition in the logarithmic sense)

The partition of the interval $[a, T]$:

$$a = t_0 < t_1 < \cdots < t_N = T.$$

Case B : Special non-uniform partition (uniform partition in the logarithmic sense)

$$\begin{aligned} t_k &= \exp\left(\log t_0 + k\widetilde{\tau}\right), \ (\text{different nodes}) \\ \widetilde{\tau} &= \log t_k - \log t_{k-1} = \frac{\log T - \log a}{N} \ (1 \le k \le N). \end{aligned} \tag{3.1}$$

L1-2 formula with $0 < \alpha < 1$

The L1-2 formula can be rewritten as the following form under uniform division in the logarithmic sense (Case B).

$$ {}_{CH}\mathbb{D}^{\alpha}_{a,t} f^k = \frac{(\widetilde{\tau})^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} \tilde{c}^{(\alpha)}_{j,k} \left(f^j - f^{j-1}\right), \tag{3.2} $$

where

$$ \begin{aligned} \tilde{c}^{(\alpha)}_{j,k} &= \begin{cases} \tilde{a}^{(\alpha)}_{1,k} + \tilde{b}^{(\alpha)}_{2,k}, \ j = 1, \\ \tilde{a}^{(\alpha)}_{j,k} - \tilde{b}^{(\alpha)}_{j,k} + \tilde{b}^{(\alpha)}_{j+1,k}, \ 2 \le j \le k-1, \\ \tilde{a}^{(\alpha)}_{k,k} - \tilde{b}^{(\alpha)}_{k,k}, \ j = k, \end{cases} \\ \tilde{a}^{(\alpha)}_{j,k} &= (k-j+1)^{1-\alpha} - (k-j)^{1-\alpha}, \\ \tilde{b}^{(\alpha)}_{j,k} &= \frac{1}{2}\Big[(k-j)^{1-\alpha} + (k-j+1)^{1-\alpha}\Big] \\ &\quad + \frac{1}{2-\alpha}\Big[(k-j)^{2-\alpha} - (k-j+1)^{2-\alpha}\Big]. \end{aligned} \tag{3.3} $$

L1-2 formula with $0 < \alpha < 1$

Theorem

Letting $f(t) \in C^3[a,T]$ *and* $0 < \alpha < 1$, *for* $t_k = \exp(\log t_0 + k\widetilde{\tau})$ *and* $\widetilde{\tau} = \log t_k - \log t_{k-1}$ $(0 \le j \le k)$, *then the truncation errors* R^k $(1 \le k \le N)$ *in (2.7) satisfy*

$$
\begin{aligned}
|R^1| &\le \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} \left|\delta^2 f(t)\right| \widetilde{\tau}^{2-\alpha}, \\
|R^k| &\le \frac{\alpha}{8\Gamma(1-\alpha)} \max_{t_0 \le t \le t_1} |\delta^2 f(t)| \left(\log \frac{t_k}{t_1}\right)^{-1-\alpha} \widetilde{\tau}^3 \\
&\quad + \max_{t_0 \le t \le t_k} |\delta^3 f(t)| \left[\frac{1}{12\Gamma(1-\alpha)} + \frac{\alpha(5-\alpha)}{6\Gamma(4-\alpha)}\right] \widetilde{\tau}^{3-\alpha}, \quad k \ge 2.
\end{aligned}
\tag{3.4}
$$

L1-2 formula with $0 < \alpha < 1$

The coefficients obtained in this case are the same as those obtained by L1-2 formula of the Caputo derivative.

Lemma

For $\alpha \in (0,1)$, *coefficients* $\tilde{c}_{j,k}^{(\alpha)}$ $(1 \le j \le k,\ 1 \le k \le N)$ *satisfy*

(1) $k = 1:$ $\tilde{c}_{1,1}^{(\alpha)} = 1,$

(2) $k = 2:$ 1) $\tilde{c}_{1,2}^{(\alpha)} = 2^{1-\alpha} - \left(\frac{1}{2} + \frac{1}{2-\alpha}\right) \in \left(-\frac{1}{2}, 1\right),$

$\tilde{c}_{2,2}^{(\alpha)} = \frac{1}{2} + \frac{1}{2-\alpha} \in \left(1, \frac{3}{2}\right),$

2) $\left|\tilde{c}_{1,2}^{(\alpha)}\right| < \tilde{c}_{2,2}^{(\alpha)},$

(3) $k \ge 3$: 1) $\tilde{c}_{k,k}^{(\alpha)} > \left|\tilde{c}_{k-1,k}^{(\alpha)}\right|,$

2) $\tilde{c}_{k,k}^{(\alpha)} > \tilde{c}_{k-2,k}^{(\alpha)},$

3) $\tilde{c}_{j,k}^{(\alpha)} > 0,\ \ j \ne k-1,$

4) $\tilde{c}_{k-2,k}^{(\alpha)} > \tilde{c}_{k-3,k}^{(\alpha)} > \cdots > \tilde{c}_{1,k}^{(\alpha)},$

5) $\sum_{j=1}^{k} \tilde{c}_{j,k}^{(\alpha)} = k^{1-\alpha}.$

L2-1$_\sigma$ formula with $0 < \alpha < 1$

For $\sigma = 1 - \frac{\alpha}{2}$ and $t_{k+\sigma} = \exp(\log t_k + \sigma\widetilde{\tau})$, the L2-1$_\sigma$ formula on uniform partition in the logarithmic sense (Case B) can be

$$ {}_{CH}\mathfrak{D}^{\alpha}_{a,t} f^{k+\sigma} = \frac{(\widetilde{\tau})^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k+1} \tilde{c}^{(\alpha,\sigma)}_{j,k} \left(f^j - f^{j-1}\right), \tag{3.5} $$

where

$$ \tilde{c}^{(\alpha,\sigma)}_{j,k} = \begin{cases} \tilde{a}^{(\alpha,\sigma)}_{1,k} - \tilde{b}^{(\alpha,\sigma)}_{1,k}, & j = 1, \\ \tilde{a}^{(\alpha,\sigma)}_{j,k} + \tilde{b}^{(\alpha,\sigma)}_{j-1,k} - \tilde{b}^{(\alpha,\sigma)}_{j,k}, & 2 \le j \le k, \\ \tilde{b}^{(\alpha,\sigma)}_{k,k} + \sigma^{1-\alpha}, & j = k+1, \end{cases} $$

$$ \begin{aligned} \tilde{a}^{(\alpha,\sigma)}_{j,k} &= (k+\sigma-j+1)^{1-\alpha} - (k+\sigma-j)^{1-\alpha}, \\ \tilde{b}^{(\alpha,\sigma)}_{j,k} &= \frac{1}{2-\alpha}\left[(k+\sigma-j+1)^{2-\alpha} - (k+\sigma-j)^{2-\alpha}\right] \\ &\quad - \frac{1}{2}\left[(k+\sigma-j+1)^{1-\alpha} + (k+\sigma-j)^{1-\alpha}\right]. \end{aligned} \tag{3.6} $$

L2-1$_\sigma$ formula with $0 < \alpha < 1$

Theorem

Letting $f(t) \in C^3[a,T]$ *and* $\alpha \in (0,1)$, *for the fixed* $\sigma = 1 - \frac{\alpha}{2}$, *the truncation errors* $R^{k+\sigma}$ $(0 \le k \le N-1)$ *defined in (2.19) with* $t_k = \exp(\log t_0 + k\widetilde{\tau})$ *and* $t_{k+\sigma} = \exp(\log t_0 + (k+\sigma)\widetilde{\tau})$ *satisfy*

$$\left|R^{k+\sigma}\right| \le \left\{ \frac{\sigma^{-\alpha}}{12\Gamma(1-\alpha)} \max_{t_0 \le t \le t_{k+1}} \left|\delta^3 f(t)\right| + \frac{\sigma^{1-\alpha}}{6\Gamma(2-\alpha)} \max_{t_k \le t \le t_{k+1}} \left|\delta^3 f(t)\right| \right\} \widetilde{\tau}^{3-\alpha}. \tag{3.7}$$

L2-1$_\sigma$ formula with $0 < \alpha < 1$

The coefficients $\tilde{c}_{j,k}^{(\alpha,\sigma)}$ in (3.6) are identical to the coefficients of L2-1$_\sigma$ formula of the Caputo derivative which satisfy the following properties.

Lemma

For any order $\alpha \in (0,1)$, $\sigma = 1 - \frac{\alpha}{2}$ *and* $t_j = \exp\left(\log t_0 + j\tilde{\tau}\right)$ $(0 \le j \le k+1)$, *coefficients* $\tilde{c}_{j,k}^{(\alpha,\sigma)}$ $(1 \le j \le k+1)$ *in (3.6) satisfy*

$(1)\ \tilde{c}_{j,k}^{(\alpha,\sigma)} > \frac{1-\alpha}{2}\left(k-j+1+\sigma\right)^{-\alpha}$,

$(2)\ \tilde{c}_{k+1,k}^{(\alpha,\sigma)} > \tilde{c}_{k,k}^{(\alpha,\sigma)} > \tilde{c}_{k-1,k}^{(\alpha,\sigma)} > \cdots > \tilde{c}_{2,k}^{(\alpha,\sigma)} > \tilde{c}_{1,k}^{(\alpha,\sigma)}$,

$(3)\ \left(2\sigma-1\right)\tilde{c}_{k+1,k}^{(\alpha,\sigma)} > \sigma\tilde{c}_{k,k}^{(\alpha,\sigma)}$.

H2N2 formula with $1 < \alpha < 2$

Let $t_{k-\frac{1}{2}} = \exp\left(\log t_k - \frac{1}{2}\widetilde{\tau}\right) = \sqrt{t_{k-1}t_k}$ (geometric mean). The H2N2 formula on uniform partition in the logarithmic sense (Case B) can be

$$
\begin{aligned}
&{}_{CH}\mathbb{D}_{a,t}^{\alpha} f^{k-\frac{1}{2}} \\
&= \frac{2(\widetilde{\tau})^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{k} \tilde{c}_{j,k}^{(\alpha)} \left(f^j - f^{j-1}\right) - \frac{2(\widetilde{\tau})^{1-\alpha}}{\Gamma(3-\alpha)} \tilde{a}_{0,k}^{(\alpha)}\, \delta f(t_0),
\end{aligned} \tag{3.8}
$$

where

$$
\tilde{c}_{j,k}^{(\alpha)} = \begin{cases} \tilde{a}_{0,k}^{(\alpha)} - \tilde{a}_{1,k}^{(\alpha)}, & j = 1, \\ \tilde{a}_{j-1,k}^{(\alpha)} - \tilde{a}_{j,k}^{(\alpha)}, & 2 \le j \le k-1, \\ \tilde{a}_{k-1,k}^{(\alpha)}, & j = k, \end{cases} \tag{3.9}
$$

$$
\tilde{a}_{j,k}^{(\alpha)} = \begin{cases} (k-\frac{1}{2})^{2-\alpha} - (k-1)^{2-\alpha}, & j = 0, \\ \frac{1}{2}[(k-j)^{2-\alpha} - (k-j-1)^{2-\alpha}], 1 \le j \le k-1. \end{cases}
$$

H2N2 formula with $1 < \alpha < 2$

Theorem

Supposing $f(t) \in C^3[a,T]$ *and* $\alpha \in (1,2)$*, the following inequalities for the truncation errors* $R^{k-\frac{1}{2}}$ $(1 \le k \le N)$ *defined in (2.32) with* $t_j = \exp\left(\log t_0 + j\widetilde{\tau}\right)$ *and* $\bar{t}_{j-\frac{1}{2}} = \exp\left(\log t_0 + (j-\frac{1}{2})\widetilde{\tau}\right)$ *hold*

$$|R^{k-\frac{1}{2}}| \le \frac{1}{2^{2-\alpha}\Gamma(3-\alpha)} \max_{t_0 \le t \le t_1} |\delta^3 f(t)| \widetilde{\tau}^{3-\alpha}, \;\; k = 1,$$

$$|R^{k-\frac{1}{2}}| \le \max_{t_0 \le t \le t_k} |\delta^3 f(t)| \left\{ \frac{5}{3\Gamma(2-\alpha)} + \frac{1}{\Gamma(3-\alpha)} \right\} \widetilde{\tau}^{3-\alpha}, k \ge 2. \tag{3.10}$$

H2N2 formula with $1 < \alpha < 2$

The coefficients $\tilde{c}_{j,k}^{(\alpha)}$ are similar to the coefficients of the H2N2 formula of the Caputo derivative.

Lemma

For coefficients $\tilde{c}_{j,k}^{(\alpha)}$ *defined in (3.9) with* $t_j = \exp\left(\log t_0 + j\tilde{\tau}\right)$ *and* $\bar{t}_{j-\frac{1}{2}} = \exp\left(\log t_0 + (j-\frac{1}{2})\tilde{\tau}\right)$ $(1 \le j \le k,\ 1 \le k \le N)$ *and* $\alpha \in (1,2)$*, it holds that*

$$(1)\ \tilde{c}_{k,k}^{(\alpha)} > 0, \quad \tilde{c}_{j,k}^{(\alpha)} < 0\,(1 \le j \le k-1),$$

$$(2)\ \tilde{c}_{1,k}^{(\alpha)} > \tilde{c}_{2,k}^{(\alpha)} > \cdots > \tilde{c}_{k-1,k}^{(\alpha)}\,(k \ge 3),$$

$$(3)\ \left|\tilde{c}_{k-1,k}^{(\alpha)}\right| < \tilde{c}_{k,k}^{(\alpha)}.$$

Numerical examples

Example

Consider the following fractional ordinary differential equation with initial value condition and $\alpha \in (0,1)$

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a,t}u(t) = g(t), \quad t \in [a,T], \\ u(a) = u_a. \end{cases} \tag{4.1}$$

Let $a = 1$, $T = 2$, $u_a = 0$ and $g(t) = \frac{6}{\Gamma(4-\alpha)}(\log t)^{3-\alpha}$, and then the exact solution $u(t) = (\log t)^3$.

Numerical examples

Table: Errors and convergence results for L1-2 formula.

	α	0.1		0.5		0.9	
Case	N	Error	Rate	Error	Rate	Error	Rate
	20	1.2949E-05	–	1.7132E-04	–	1.2500E-03	–
A	40	1.8958E-06	2.8212	3.0394E-05	2.5392	2.8647E-04	2.1632
	80	2.7671E-07	2.8012	5.3986E-06	2.5154	6.6144E-05	2.1336
	160	4.0124E-08	2.7983	9.5901E-07	2.5041	1.5345E-05	2.1173
	20	8.8798E-06	–	1.6124E-04	–	1.0865E-03	–
B	40	1.2834E-06	2.7905	2.9073E-05	2.4715	2.5424E-04	2.0953
	80	1.8364E-07	2.8050	5.2094E-06	2.4804	5.9408E-05	2.0974
	160	2.6063E-08	2.8167	9.2956E-07	2.4865	1.3871E-05	2.0986

Numerical examples

Table: Errors and convergence results for L2-1_σ formula.

	α	0.1		0.5		0.9	
Case	N	Error	Rate	Error	Rate	Error	Rate
	20	3.9732E-06	–	2.7454E-05	–	6.9885E-05	–
A	40	6.9995E-07	2.5495	5.5778E-06	2.3401	1.6838E-05	2.0898
	80	1.1635E-07	2.6119	1.0820E-06	2.3872	4.0016E-06	2.0916
	160	1.8571E-08	2.6592	2.0379E-07	2.4192	9.4325E-07	2.0942
	20	4.4234E-06	–	4.3617E-05	–	1.5784E-04	–
B	40	6.8545E-07	2.6900	8.1995E-06	2.4112	3.7218E-05	2.0844
	80	1.0347E-07	2.7278	1.5124E-06	2.4386	8.7374E-06	2.0907
	160	1.5318E-08	2.7558	2.7539E-07	2.4573	2.0458E-06	2.0945

Numerical examples

Example

For $\alpha \in (1,2)$, we consider the fractional initial value problem

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a,t}u(t) = g(t), \quad t \in [a,T], \\ u(a) = u_a, \ \delta u(a) = v_a. \end{cases} \tag{4.2}$$

Let $a = 1$, $T = 2$, $u_a = v_a = 0$ and $g(t) = \frac{6}{\Gamma(4-\alpha)}(\log t)^{3-\alpha}$, so we can derive the exact solution $u(t) = (\log t)^3$.

Numerical examples

Table: Errors and convergence results for H2N2 formula

	α	1.2		1.5		1.8	
Case	N	Error	Rate	Error	Rate	Error	Rate
	50	1.7184E-04	–	1.0856E-03	–	4.7019E-03	–
A	100	5.2463E-05	1.7239	3.9587E-04	1.4658	2.0830E-03	1.1830
	150	2.6012E-05	1.7373	2.1811E-04	1.4761	1.2886E-03	1.1893
	200	1.5773E-05	1.7438	1.4262E-04	1.4808	9.1538E-04	1.1920
	50	1.0626E-04	–	8.8392E-04	–	3.9965E-03	–
B	100	3.4447E-05	1.6251	3.2516E-04	1.4427	1.7692E-03	1.1756
	150	1.7577E-05	1.6593	1.7992E-04	1.4595	1.0943E-03	1.1848
	200	1.0854E-05	1.6757	1.1797E-04	1.4671	7.7736E-04	1.1887

Lorenz system

Consider the following Lorenz system with Caputo-Hadamard fractional derivative with $\alpha \in (0,1)$

$$\begin{cases} {}_{CH}\mathrm{D}_{a,t}^{\alpha} x_1(t) = \bar{a}(x_2(t) - x_1(t)), \\ {}_{CH}\mathrm{D}_{a,t}^{\alpha} x_2(t) = \bar{c}x_1(t) - x_2(t) - x_1(t)x_3(t), \\ {}_{CH}\mathrm{D}_{a,t}^{\alpha} x_3(t) = x_1(t)x_2(t) - \bar{b}x_3(t), \end{cases} \tag{5.1}$$

where $t > a > 0$, $\bar{a}$, $\bar{b}$ and $\bar{c}$ are intrinsic parameters. For the given parameter value $(\bar{a}, \bar{b}, \bar{c}) = (10, \frac{8}{3}, 200)$, we choose the initial value $(x_1(a), x_2(a), x_3(a)) = (x_1(2.5), x_2(2.5), x_3(2.5)) = (5, 3, 9)$, $t \in [a, T] = [2.5, T]$, $T = 60$.

Lorenz system

Table: The existence of chaotic attractors with changed α.

α	$\max\{l_1, l_2, l_3\}$	Existence of chaotic attractor
1.00000	1.374808979490	yes
0.95000	0.805883701541	yes
0.93750	0.362285530845	yes
0.93125	2.686635511486	yes
0.92500	-1.214511641728	no
0.90000	-1.917961361415	no

Lorenz system

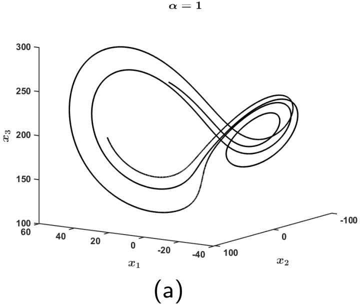

Figure: The chaotic attractor of system (5.1) using L1-2 method.

Lorenz system

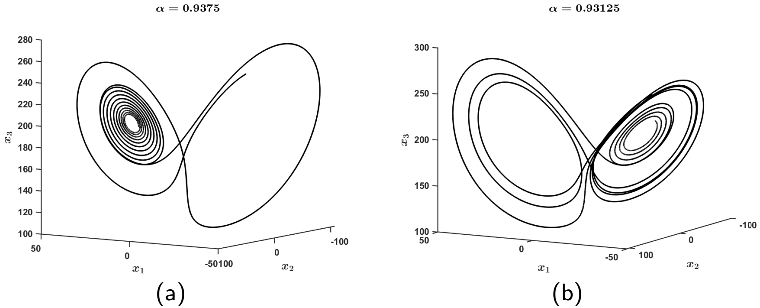

Figure: The chaotic attractor of system (5.1) using L1-2 method.

Lorenz system

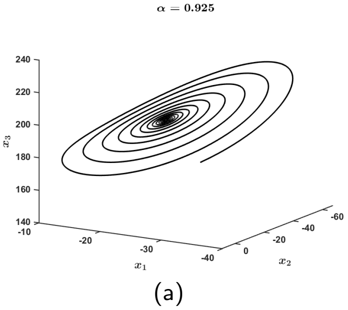

Figure: The phase portrait of system (5.1) using L1-2 method. No chaotic attractors in this case.

Bagley-Torvik system

Let us consider the following Bagley-Torvik problem with Caputo-Hadamard fractional derivative of order $\alpha = 3/2$,

$$\begin{cases} A\dfrac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + B\,{}_{CH}\mathrm{D}_{1,t}^{3/2} y(t) + Cy(t) = g(t), \ t > 1, \\ y(1) = 0, \ \delta y(t)|_{t=1} = 0, \end{cases} \tag{5.2}$$

where parameters A, B and C are constants. For numerical calculation, choose $T = 100$, $A = 1$, $B = C = 0.5$ and the source term

$$g(t) = \begin{cases} 8, & 1 \leq t \leq 2, \\ 0, & (100 = T \geq) \, t > 2. \end{cases} \tag{5.3}$$

Bagley-Torvik system

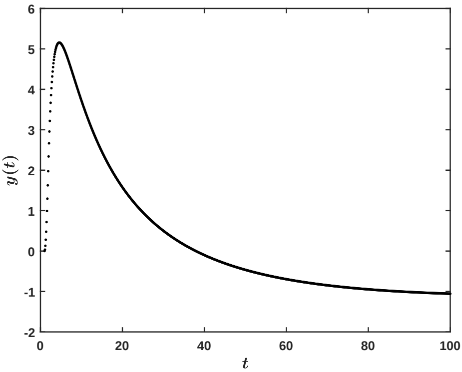

Figure: The solutions of Bagley-Torvik system by H2N2 formula.

Acknowledgement

Thank you all for your attention!!!