Sharp pointwise-in-time error estimate of L1 scheme for nonlinear subdiffusion equations

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Outline

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Background

- nonlinear subdiffusion equations

$$\partial_t^\alpha u - \Delta u = f(u), \quad x \in \Omega \times (0, T],$$

where

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^\alpha} ds, \quad 0 < \alpha < 1.$$

- physical view
  - quantum physics: describe non-Markovian evolution
  - quantum mechanics : describe quantum dynamics
- Mathematical view
  - based on the quantum energy
  - the fractional path integral and fractional Brownian motion
  - using a fractional variational principle

Regularity of the solutions

- If the initial condition $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, the solution satisfies

$$\|\partial_t u(t)\|_{L^2(\Omega)} \leq Ct^{\alpha-1}$$

(B. Jin, B. Li, Z. Zhou, SIAM J. Numer. Anal,2018.)

- If $u_0 \in \dot{H}^\nu(\Omega)$ with $\nu \in (0, 2]$ and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, the solution satisfies

$$\|\partial_t u(t)\|_{L^2(\Omega)} \leq Ct^{\nu\alpha/2-1}$$

(M. Maskari, S. Karaa, SIAM J. Numer. Anal, 2019.)

- smooth initial conditions $+$ f+compatibility conditions: smooth solutions

Present results

Under the assumption

$$\|\partial_t u(t)\|_{L^2(\Omega)} \leq Ct^{\alpha-1},$$

many works indicate that the errors of many schemes in the maximum norm are $\mathcal{O}(\tau^\alpha)$

- L1 schemes on the uniform meshes
- L2 schemes on the uniform meshes
- Convolution quadrature Euler method
- convolution quadrature BDF methods

Numerical simulations show an interesting phenomenon that the convergence order of the schemes is $\mathcal{O}(\tau^\alpha)$ as t tends to 0, and high-order accurate at the final time $t = T$.

Present the pointwise error estimates

Linear problem

$$\partial_t^\alpha u = \Delta u \quad x \in \Omega \times (0, T],$$

- L1 scheme on the uniform meshes, $\tau t_n^{\alpha-1}$, Gracia-ORiordan-Stynes (CMAM 2018)
- A modified L1 scheme, $\tau t_n^{\alpha-1}$, Yan-Khan-Ford (SINUM 2018)
- L1 and L2 scheme on graded and uniform meshes, N. Kopteva, X. Meng, (MC 2019, 2021)
- Correction Convolution quadrature BDF method, $\tau^k t_n^{\alpha-k}$ Jin-Li-Zhou (SISC 2017)
- Grünwald – Letnikov scheme, $\tau t_n^{\alpha-1}$, Chen-Holland-Stynes, (APNUM 2019)
- $\cdots$

Present the pointwise error estimates

Semi-linear problem

$$\partial_t^\alpha u = \Delta u + f(u) \quad x \in \Omega \times (0, T],$$

- Convolution quadrature Euler method and assumption $\|\partial_t u(t)\|_{L^2(\Omega)} \le Ct^{\nu\alpha/2-1}$, Maskari-Karaa (SINUM 2019) a generalized variant of the standard discrete Gronwall’ s inequality
- L1 scheme on the graded meshes and assumption $\|\partial_t u(t)\|_{L^2(\Omega)} \le Ct^{\alpha-1}$, N. Kopteva, (SINUM 2020) Some assumption on the nonlinear function f are needed. The proposed conditions on the nonlinearity can guarantee the exact solutions have the upper and lower bounds, and a method of upper and lower solutions is introduced to address that the numerical solutions lie within a certain range.
- Convolution quadrature BDF methods and assumption $u_0 \in H_0^1(\Omega) \cap C^2(\bar{\Omega})$

motivation

Semi-linear problem

$$\partial_t^\alpha u = \Delta u + f(u) \quad x \in \Omega \times (0, T],$$

Our motivation

- a more generalized nonlinear function f?
- a more generalized assumption on the regularity of the solutions?
- L1 scheme, Convolution quadrature BDF methods, Grünwald – Letnikov scheme and the other scheme?

A framework of pointwise-in-time error estimates of numerical schemes

Assumption

- We assume that

$$\|\partial_t^m u\|_{L^2(\Omega)} \le C t^{\sigma - m}, \qquad \text{for } m = 1, 2, \text{ and } \sigma \in (0,1) \cup (1,2].$$

- $f(u) \in C^2(\mathbb{R})$

Remark: The assumption $\sigma \in (0, \alpha]$ is reasonable and widely accepted. Suppose that the solution is smoother, i.e., $\sigma > \alpha$, some additional hypothesis should be added. In this work, we assume that $\sigma \in (0,1) \cup (1,2)$ in order to make the current analysis extendable.

Linearized numerical methods

- central finite difference method

$$\sum_{j=1}^{d} u^n_{x_jx_j} = \frac{1}{h^2}\sum_{j=1}^{d}(u^n_{i_1,\cdots,x_j-1,\cdots,i_d} - 2u^n_{i_1,\cdots,i_d} + u^n_{i_1,\cdots,x_j+1,\cdots,i_d}) + R^n_{i_1,\cdots,i_d}$$

$$:= \sum_{j=1}^{d} \delta^2_{x_j} u^n_{i_1,\cdots,i_d} + R^n_{i_1,\cdots,i_d}, \qquad (2.1)$$

- standard L1-approximation

$$\partial^{\alpha}_{t_n} u = \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{n}\frac{u^j_{i_1,\cdots,i_d} - u^{j-1}_{i_1,\cdots,i_d}}{\tau}\int_{t_{j-1}}^{t_j}\frac{1}{(t_n-s)^{\alpha}}ds + r^n_{i_1,\cdots,i_d}$$

$$= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{n} a_{n-j}(u^j_{i_1,\cdots,i_d} - u^{j-1}_{i_1,\cdots,i_d}) + r^n_{i_1,\cdots,i_d}$$

$$:= D^{\alpha}_{\tau} u^n_{i_1,\cdots,i_d} + r^n_{i_1,\cdots,i_d}, \qquad (2.2)$$

Linearized numerical methods

- Newton linearized method

$$\begin{aligned} f(u^n_{i_1,\cdots,i_d}) &= f(u^{n-1}_{i_1,\cdots,i_d}) + f_1(u^{n-1}_{i_1,\cdots,i_d})(u^n_{i_1,\cdots,i_d} - u^{n-1}_{i_1,\cdots,i_d}) \\ &+ \bar{r}^n_{i_1,\cdots,i_d} \end{aligned} \tag{2.3}$$

- Fully-discrete and linearized scheme

$$\begin{aligned} D^\alpha_\tau U^n_{i_1,\cdots,i_d} &= \sum_{j=1}^{d} \delta^2_{x_j} U^n_{i_1,\cdots,i_d} + f(U^{n-1}_{i_1,\cdots,i_d}) \\ &+ f_1(U^{n-1}_{i_1,\cdots,i_d})(U^n_{i_1,\cdots,i_d} - U^{n-1}_{i_1,\cdots,i_d}) \end{aligned}$$

with $U^0_{i_1,\cdots,i_d} = u^0_{i_1,\cdots,i_d}$.

main results

- Convergence

Suppose that the system has a unique solution satisfying the assumptions. Then, there exists a positive constant τ_0, such that when $\tau \leq \tau_0$, the full finite difference system admits a unique solution U^n, it holds for $n = 1, \cdots, N$ that

$$\|u^n - U^n\|_\infty \lesssim \begin{cases} \tau^{\sigma+1-\alpha} t_n^{\alpha-1} + t_n^\alpha h^2, & 0 < \sigma < 1, \\ \tau^{2-\alpha} t_n^{\alpha+\sigma-2} + t_n^\alpha h^2, & 1 < \sigma \leq 2, \end{cases}$$

where

$$\begin{aligned} u^n =& [u_{1,1,\cdots,1}, u_{2,1,\cdots,1}, \cdots, u_{M-1,1,\cdots,1}, \cdots, u_{1,1,\cdots,M-1}, \\ & u_{2,1,\cdots,M-1}, \cdots, u_{M-1,1,\cdots,M-1}]^T, \\ U^n =& [U_{1,1,\cdots,1}, U_{2,1,\cdots,1}, \cdots, U_{M-1,1,\cdots,1}, \cdots, U_{1,1,\cdots,M-1}, \\ & U_{2,1,\cdots,M-1}, \cdots, U_{M-1,1,\cdots,M-1}]^T. \end{aligned}$$

Some remarks

Remark 1

- The convergence results imply that, when $t_n \to 0$, it holds that

$$\max_{1\le n\le N} \|u^n - U^n\|_\infty \lesssim \tau^\sigma + \tau^\alpha h^2.$$

When t is far away form 0, it holds that

$$\|u^n - U^n\|_\infty \lesssim \begin{cases} \tau^{\sigma+1-\alpha} + h^2, & \sigma \in (0,1), \\ \tau^{2-\alpha} + h^2, & \sigma \in (1,2]. \end{cases}$$

Some remarks

Remark 2

- If we consider the maximum error in the whole domain $\Omega \times [0, T]$, we have

$$\max_{1 \le n \le N} \|u^n - U^n\|_\infty \lesssim \begin{cases} \tau^\sigma + h^2, & \sigma \in (0,1) \cup (1, 2-\alpha), \\ \tau^{2-\alpha} + h^2, & \sigma \in [2-\alpha, 2]. \end{cases}$$

- This convergent results not only consistent with the previous pointwise error estimates for L1-scheme under $\sigma = \alpha$, but also give the new pointwise error estimate for $\alpha \neq \sigma$.

Proof of our main results

- truncation errors of nonlinear item
  Suppose that u satisfies the solution regularity. Then, it holds that

$$|r_n| := \left|D_\tau^\alpha u^n - \partial_{t_n}^\alpha u\right| \lesssim \begin{cases} \tau^{\sigma-\alpha} n^{-\min(1+\alpha, 2-\sigma)}, & 0 < \sigma < 1, \\ \tau^{\sigma-\alpha} n^{-2+\sigma}, & 1 < \sigma < 2. \end{cases}$$

Proof of our main results

- Discrete fractional Grönwall inequality
  Suppose $0 < \alpha < 1$ and $\tau > 0$. Let y_i, $0 \leq i \leq N$, be a sequence of non-negative real numbers satisfying

$$D_\tau^\alpha y_n \leq \lambda y_n + \lambda y_{n-1} + \mu_1 n^{-\sigma_1} + \mu_2 n^{-\sigma_2} + \eta, \text{ for } n = 1, \ldots, N,$$

where $\sigma_1 > 1, \sigma_2 < 1, \lambda > 0$, and $\mu_1, \mu_2 \geq 0$. Then there exists a constant $\tau_* = \sqrt[\alpha]{\frac{1}{2\Gamma(2-\alpha)\lambda}}$ such that $\tau < \tau^*$, it holds

$$\begin{aligned} y_n \leq\ & 2\Gamma(2-\alpha)C_*(1 + E_{\alpha,1}(2\Gamma(2-\alpha)\lambda_1 t_n^\alpha))\mu_1 \tau {t_n}^{\alpha-1} \\ & + \Gamma(1-\sigma_2)2\Gamma(2-\alpha)C_*(1 + E_{\alpha,1-\sigma_2+\alpha}(2\Gamma(2-\alpha)\lambda_1 t_n^\alpha))\mu_2 \tau^{\sigma_2} t_n^{\alpha-\sigma_2} \\ & + 2\Gamma(2-\alpha)C_*(1 + E_{\alpha,1+\alpha}(2\Gamma(2-\alpha)\lambda_1 t_n^\alpha))(4y_0 + \eta)t_n^\alpha, \end{aligned}$$

where $\lambda_1 = \frac{(3-2^\alpha)}{2-2^\alpha}\lambda$, $t_n = n\tau$ and C_* is a constant.

Proof of our main results

- Set $\|e^n\|_\infty = e^n_{i0}$ and $e^n_i = u^n_i - U^i_n$. The error at the grid point (x_{i_0}, t_n) satisfies

$$D^\alpha_\tau e^n_{i0} = \delta^2_x e^n_{i0} + (R_f)^n_{i0} + r^n_{i0} + \bar{r}^n_{i0} + R^n_{i0}.$$

- Using the fact that $a_i > a_{i+1}$, we further have

$$\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} a_0\right)|e^n_{i0}| \quad \le \quad \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{n-1} (a_{n-j-1} - a_{n-j})|e^j_{i0}| + |(R_f)^n_{i0}$$

The above can be rewritten as

$$D^\alpha_\tau |e^n_{i0}| \le |(R_f)^n_{i0}| + |r^n_{i0}| + |\bar{r}^n_{i0}| + |R^n_{i0}|. \qquad (3.1)$$

- Using the previous lemmas, we finally get the convergence results.

Example 1

One-dimensional nonlinear subdiffusion problems

$$\partial_t^\alpha u = u_{xx} + \sqrt{1+u^2} + g(x,t), \quad (x,t) \in (0,\pi) \times (0,t_N],$$

where the initial condition and $g(x,t)$ are specially chosen such that the problem admits an exact solution in the form of

$$u(x,t) = t^\sigma \sin(x).$$

Numerical result

- Maximum errors at $t = 1$ and convergence orders in temporal direction with $M = 1000$ for Example 1

α	$\sigma \backslash N$	10	20	40	80	160	$Rate$	expected order
0.4	0.1	3.50e-2	2.05e-2	1.22e-2	7.37e-3	4.46e-3	0.73	$\sigma+1-\alpha$
	0.4	1.12e-2	5.21e-3	2.46e-3	1.18e-3	5.71e-4	1.07	$\sigma+1-\alpha$
	0.6	5.13e-3	2.10e-3	8.70e-4	3.63e-4	1.53e-4	1.25	$\sigma+1-\alpha$
	1.2	3.98e-3	1.57e-3	6.17e-4	2.41e-4	9.40e-5	1.36	$2-\alpha$
	1.8	1.30e-2	5.03e-3	1.93e-3	7.38e-4	2.81e-4	1.39	$2-\alpha$
0.6	0.4	3.27e-2	1.78e-2	9.89e-3	5.54e-3	3.13e-3	0.84	$\sigma+1-\alpha$
	0.6	1.51e-2	7.28e-3	3.54e-3	1.73e-3	8.51e-4	1.03	$\sigma+1-\alpha$
	0.8	5.68e-3	2.49e-3	1.09e-3	4.76e-4	2.08e-4	1.19	$\sigma+1-\alpha$
	1.2	3.98e-3	1.57e-3	6.17e-4	2.41e-4	9.40e-5	1.36	$2-\alpha$
	1.8	1.30e-2	5.03e-3	1.93e-3	7.38e-4	2.81e-4	1.39	$2-\alpha$

Numerical result

- Maximum errors as $t \to 0$ and convergence orders in temporal direction with $M = 1000$ and $N = 10$ for Example 1

α	$\sigma \setminus t_N$	$1e-3$	$1e-4$	$1e-5$	$1e-6$	$1e-7$	$Rate$	expected order
0.4	0.1	2.90e-2	2.39e-2	1.92e-2	1.54e-2	1.23e-2	0.09	σ
	0.4	1.20e-3	5.02e-4	2.05e-4	8.22e-5	3.28e-5	0.40	σ
	0.6	1.38e-4	3.67e-5	9.43e-6	2.39e-6	6.03e-7	0.59	σ
	1.2	5.61e-7	3.70e-8	2.38e-9	1.51e-10	9.57e-12	1.19	σ
	1.8	2.87e-8	4.71e-10	7.58e-12	1.21e-13	1.92e-15	1.80	σ
0.6	0.4	3.67e-3	1.49e-3	5.94e-4	2.37e-4	9.42e-5	0.40	σ
	0.6	4.29e-4	1.09e-4	2.76e-5	6.95e-6	1.75e-6	0.60	σ
	0.8	3.99e-5	6.43e-6	1.02e-6	1.62e-7	2.57e-8	0.80	σ
	1.2	1.64e-6	1.05e-7	6.64e-9	4.49e-10	2.55e-11	1.20	σ
	1.8	7.58e-7	1.21e-9	1.92e-11	3.05e-13	4.85e-14	1.80	σ

Numerical result

- Maximum errors at $t = 1$ and convergence orders in spatial direction with $N = 1000$ for Example 1

α	$\sigma \backslash M$	8	16	24	32	40	$Rate$	expected order
0.4	0.4	9.25e-3	2.30e-3	1.02e-3	5.67e-4	3.60e-4	2.02	2
	1.2	7.68e-3	1.92E-3	8.51e-4	4.79e-4	3.06e-4	2.00	2
0.6	0.6	8.20e-3	2.03e-3	8.96e-4	4.98e-4	3.14e-4	2.03	2
	0.2	6.79e-3	1.70e-3	7.54e-4	4.24e-4	2.71e-4	2.00	2

Example 2

Consider the nonlinear subdiffusion problems

$$\partial_t^\alpha u - \Delta u = f(u), \quad x \in \Omega \times (0, T]$$

with the following conditions

$$\begin{aligned}
&(a)\ d = 1, f(u) = \sqrt{1+u^2} \text{ and } u_0(x) = x(1-x), \Omega = (0,1),\\
&(b)\ d = 1, f(u) = u - u^3 \text{ and } u_0(x) = \sin(\pi x), \Omega = (0,1),\\
&(c)\ d = 2, f(u) = \sqrt{1+u^2} \text{ and } u_0(x) = \sin(\pi x)\sin(\pi y), \Omega = (0,1)^2,\\
&(d)\ d = 2, f(u) = u - u^3 \text{ and } u_0(x) = x(1-x)y(1-y), \Omega = (0,\pi)^2.
\end{aligned}$$

The reference solutions are obtained by using small temporal stepsizes.

Numerical result

- Maximum errors at $t = 1$ and convergence orders in temporal direction for Example 2

α	$case\backslash N$	10	20	40	80	160	$Rate$	expected order
0.4	(a)	1.74e-4	8.36e-5	4.04e-5	1.94e-5	8.99e-6	1.06	1
	(b)	1.49e-3	7.14e-4	3.46e-4	1.66e-4	7.69e-5	1.06	1
	(c)	9.63e-5	4.77e-5	2.36e-5	1.16e-5	5.61e-6	1.03	1
	(d)	7.20e-6	3.57e-6	1.76e-6	8.67e-7	4.19e-7	1.02	1
0.6	(a)	2.15e-4	1.01e-4	4.84e-5	2.30e-5	1.06e-5	1.08	1
	(b)	1.88e-3	8.84e-4	4.23e-4	2.01e-4	9.27e-5	1.08	1
	(c)	1.06e-4	5.21e-5	2.57e-5	1.26e-5	6.09e-6	1.03	1
	(d)	7.94e-6	3.92e-6	1.93e-6	9.47e-7	4.57e-7	1.03	1
0.8	(a)	2.13e-4	9.53e-5	4.44e-5	2.08e-5	9.48e-6	1.10	1
	(b)	1.96e-3	8.74e-4	4.07e-5	1.90e-5	8.66e-6	1.12	1
	(c)	7.91e-5	3.88e-5	1.91e-5	9.31e-6	4.47e-6	1.04	1
	(d)	6.02e-6	2.95e-6	1.45e-6	7.06e-7	3.40e-7	1.03	1

Numerical result

- Maximum errors as $t \to 0$ and convergence orders in temporal direction for Example 2

α	$case \backslash t_N$	1e-4	1e-5	1e-6	1e-7	1e-8	$Rate$	expected order
0.4	(a)	4.70e-4	2.21e-5	9.00e-5	3.61e-5	1.45e-5	0.38	α
	(b)	3.65e-3	1.87e-3	8.36e-4	3.50e-4	1.43e-4	0.35	α
	(c)	4.96e-3	3.05e-3	1.47e-3	6.34e-4	2.61e-4	0.32	α
	(d)	3.31e-4	1.87e-4	8.26e-5	3.39e-5	1.37e-5	0.35	α
0.6	(a)	1.20e-4	3.05e-5	7.70e-6	1.94e-6	4.87e-7	0.60	α
	(b)	1.14e-3	2.99e-4	7.59e-5	1.91e-5	4.81e-6	0.59	α
	(c)	2.03e-3	5.46e-4	1.39e-4	3.52e-5	8.84e-6	0.59	α
	(d)	1.11e-4	2.86e-5	7.23e-6	1.82e-6	4.57e-7	0.60	α
0.8	(a)	1.73e-5	2.75e-6	4.35e-7	6.86e-8	1.09e-8	0.80	α
	(b)	1.63e-4	2.59e-5	4.10e-6	6.51e-7	1.03e-8	0.80	α
	(c)	3.06e-4	4.77e-5	7.56e-6	1.20e-6	1.90e-7	0.80	α
	(d)	1.55e-5	2.46e-6	3.90e-7	6.18e-8	9.80e-9	0.80	α

Conclusion

- Our result
  - establish a refine DFGI, take the weak regularity of solution into account
  - at $t = T$ the convergence order is $\mathcal{O}(\tau^{\sigma+1-\alpha})$, while $0 < \sigma < 1$.
  - at $t = T$ the convergence order is $\mathcal{O}(\tau^{2-\alpha})$, while $\sigma > 1$
- Some open problems
  - the pointwise error estimate on general nonuniform steps, under general assumption on nonlinearity or $f(u) = \kappa u$ with $\kappa > 0$
  - A refined and quantitative analysis of DCC kernels on general nonuniform steps.
  - $\cdots$

Conclusion

Our recent results can be found in

- Dongfang Li, Hongyu Qin, Jiwei Zhang, Sharp pointwise-in-time error estimate of L1 scheme for nonlinear subdiffusion equations, J. Comput. Math, accepted.
- Dongfang Li, Mianfu She, Hai-wei Sun, A Novel Discrete Fractional Grönwall-Type Inequality and Its Application in Pointwise-in-Time Error Estimates, J. Sci. Comput. 2022.

Thank you!

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