Error estimation of a discontinuous Galerkin method for time fractional subdiffusion problems with nonsmooth data

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Outline

Model problem

We consider the time fractional subdiffusion equation

$$\begin{cases} u' - \mathrm{D}_{0+}^{1-\alpha} \Delta u = f & \text{in } \Omega_T := \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where

- $\alpha \in (0,1)$,
- $T > 0$ denotes the final time,
- $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ is a convex polyhedral domain,
- f and u_0 are given data, and
- $\mathrm{D}_{0+}^{1-\alpha}$ is a left-sided Riemann–Liouville fractional differential operator.

Numerical methods

There are mainly two types of methods, according to how the time fractional derivative is approximated.

- The first type of schemes are based on finite difference formula, including L-type schemes [Langlands-Henry 2005, Zhuang-Liu-Anh-Turner 2008, Gao-Sun 2011, Li-Wang-X 2021, ...] and Grünwald–Letnikov (GL) scheme [Yuste-Acedo 2005, Yuste 2006, Mohebbi-Abbaszadeh-Dehghan 2013, ...]
    - The L1 method has the accuracy $O(\tau^{1+\alpha})$ for C^2 solutions.
    - Gao-Sun-Sun 2015: some finite difference schemes of accuracy $O(\tau^2)$ for C^3 solutions, by the superconvergence property at some particular points of the GL formula.
- ...

- The second type of schemes adopt time-stepping DG methods, with graded temporal grids to conquer the singularity.
  - McLean-Mustapha 2009: piecewise constant DG method for (1.1), proved the error bound $O(\tau + |\ln\tau| h^2)$ under $L^\infty(0,T;L^2(\Omega))$-norm, with initial data $u_0 \in \dot{H}^2(\Omega)$ and the following regularity assumptions:

$$\begin{aligned} \|u(t)\|_{\dot{H}^2(\Omega)} + t\|u'(t)\|_{\dot{H}^2(\Omega)} &\leqslant M & 0 < t \leqslant T, \\ t^{2-\alpha}\|u'(t)\|_{\dot{H}^2(\Omega)} + t^{3-\alpha}\|u''(t)\|_{\dot{H}^2(\Omega)} &\leqslant M t^{\sigma-1} & 0 < t \leqslant T, \end{aligned} \tag{1.2}$$

    where σ and M are two positive constants.
  - More works using piecewise linear DG method:
    - Mustapha-McLean 2011: proved that the temporal convergence order $O(\tau^{1+\alpha})$ under the $L^\infty(0,T;L^2(\Omega))$-norm;
    - Mustapha-McLean 2012: derived the improved bound $O(\tau^{\min\{1.5+\alpha,\,2\}})$;
    - Mustapha-McLean 2013: proved the rate $O(|\ln\tau|\tau^{1+2\alpha})$, which yields superconvergence if $\alpha \in (1/2, 1)$.
    - The analyses in [Mustapha-McLean 2012, 2013] require stronger growth assumptions than (1.2).
  - Mustapha 2015: hp-version DG method for (1.1), suboptimal convergence $O(\tau^{\max\{k,2\}+(1-\alpha)/2})$, where $k \geqslant 1$ is the polynomial degree.

It is worth to noticing an alternative form of (1.1):

$$\begin{cases} \mathrm{D}_{0+}^{\alpha}(u-u_0) - \Delta u = \widetilde{f} & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $\widetilde{f} = \mathrm{I}_{0+}^{1-\alpha} f$ with $\mathrm{I}_{0+}^{1-\alpha}$ being a left-sided Riemann–Liouville fractional integral operator.

- For $f = 0$, (1.1) and (1.3) share the same solution that can be represented by the Mittag-Leffler function.
- For solution regularity and numerical analysis of problem (1.3), especially for nonsmooth data, see Ford-Xiao-Yan 2011, Li-Luo-X 2019, Li-Wang-X 2020a,2020b, Li-X 2019, Li-Yan 2018, Mustapha-Abdallah-Furati 2014, Stynes 2016,Stynes-ORiordan-Gracia 2017, Wang-Yan-Yang 2020,Yan-Khan-Ford 2018,Yang-Yan-Ford 2018,...

Regularity and growth estimates for (1.1)

- McLean 2010, McLean-Thomee 2010IMA, 2010JIEA: used Laplace transform.
- No work to investigate the weak solution to (1.1) by variational approach.
- Our work:
  - For the case $u_0 = 0,\ f \neq 0$, we introduce a weak solution to problem (1.1) via variational formulation, and prove that if $f \in L^2(0, T; \dot{H}^{-\beta}(\Omega))$ with $0 \leqslant \beta \leqslant 1$, then

$$\|u\|_{{}_0H^1(0,T;\dot{H}^{-\beta}(\Omega))} + \|u\|_{{}_0H^{1-\alpha}(0,T;\dot{H}^{2-\beta}(\Omega))} \leqslant C_\alpha \|f\|_{L^2(0,T;\dot{H}^{-\beta}(\Omega))}.$$

  - For the homogeneous case: $f = 0,\ u_0 \neq 0$, the weak solution is introduced and analyzed by Mittag-Leffler function.
  - Here we note that, instead of proving the growth estimate like (1.2), we show what kind of vector-valued Sobolev space the weak solution belongs to.

Error estimation with low regularity

- The error analyses of most existing numerical methods require either smooth property or growth estimate of the true solution.
- More challenging to establish error estimates with given low regularity data.
- A few works that aim to fill in this gap:
  - McLean-Mustapha 2015: for a temporal semi-discretization with $f = 0$, used the Laplace transform to derive

$$\|(u - u_\tau)(t_j)\|_{L^2(\Omega)} \lesssim t_j^{-1} \tau \|u_0\|_{L^2(\Omega)},$$

  - Karaa-Mustapha-Pani 2018: for a spatial semi-discretization, used the energy argument to prove that

$$\|(u-u_h)(t)\|_{L^2(\Omega)} \lesssim h^2 t^{-\alpha(2-\delta)/2} \left(\|u_0\|_{\dot{H}^\delta(\Omega)} + \sum_{i=0}^{2} \int_0^T t^i \|f^{(i)}(t)\|_{\dot{H}^\delta(\Omega)} \mathrm{d}t \right),$$

    where $0 < t \leqslant T$ and $0 \leqslant \delta \leqslant 2$.

- ...

- Our work: error estimates for a piecewise constant DG method, with nonsmooth data:
  - if $u_0 = 0$ and $f \in L^2(\Omega_T)$, then

$$(\tau^{1/2}+h)\|u-U\|_{H^{\frac{1-\alpha}{2}}(0,T;\dot{H}^1(\Omega))}+\|u-U\|_{L^2(\Omega_T)} \lesssim (\tau+h^2)\|f\|_{L^2(\Omega_T)}, \tag{1.4}$$

    optimal with respect to the solution regularity.
  - if $f = 0$ and $u_0 \in L^2(\Omega)$, then

$$\|u-U\|_{L^2(\Omega_T)} \lesssim (\tau^{1/2}+h)\|u_0\|_{L^2(\Omega)}, \tag{1.5}$$

    optimal only for temporal discretization.
- Moreover, for the case $u_0 = 0$ with uniform temporal grid, by means of Laplace transform, we prove the following quasi-optimal results:
  - if $f \in L^2(\Omega_T)$, then

$$\|u-U\|_{L^\infty(0,T;L^2(\Omega))} \lesssim |\ln \tau| \left(\tau^{1/2} + \epsilon_h h^{\min\{2,1/\alpha\}}\right) \|f\|_{L^2(\Omega_T)},$$

    where $\epsilon_h = 1$ if $\alpha \neq 1/2$ and $\epsilon_h = \sqrt{|\ln h|}$ if $\alpha = 1/2$;
  - if $f \in {}_0H^{1/2}(0,T;L^2(\Omega))$, then

$$\|u-U\|_{L^\infty(0,T;L^2(\Omega))} \lesssim |\ln \tau| \left(|\ln \tau|\tau + h^2\right) \|f\|_{{}_0H^{1/2}(0,T;L^2(\Omega))}.$$

Notation

- For a Lebesgue measurable subset ω of $\mathbb{R}^l$ ($l = 1, 2, 3$), we use $H^\gamma(\omega)$ ($\gamma \in \mathbb{R}$) and $H_0^\gamma(\omega)$ ($\gamma > 0$) to denote the standard Sobolev spaces.
- For a Lebesgue measurable subset $\mathcal{O}$ of $\mathbb{R}^l$ ($l = 1, 2, 3, 4$), the symbol $\langle p, q\rangle_{\mathcal{O}}$ means $\int_{\mathcal{O}} pq$ for $pq \in L^1(\mathcal{O})$.
- If X is a Banach space, then X^* denotes its dual space and $\langle \cdot, \cdot \rangle_X$ is the duality pairing between X^* and X.
- For $0 < \theta < 1$ and two Banach spaces X and Y, $[X, Y]_{\theta,2}$ stands for the interpolation space constructed via the K-method [Tartar 2007], with the norm

$$\|v\|_{[X,Y]_{\theta,2}} := \left(\int_0^\infty \left(t^{-\theta} K(t,v)\right)^2 \frac{\mathrm{d}t}{t}\right)^{1/2} \quad \forall\, v \in [X, Y]_{\theta,2}, \tag{2.1}$$

  where the functional $K : (0, \infty) \times (X + Y) \to \mathbb{R}$ is defined by

$$K(t, v) := \inf_{\substack{v = v_0 + v_1 \\ v_0 \in X,\, v_1 \in Y}} \{\|v_0\|_X + t\|v_1\|_Y\}.$$

- Moreover, if the symbol C has subscript(s), then it means a positive constant that depends only on its subscript(s), and its value may differ at each of its occurrence(s).

Space $\dot{H}^\gamma(\Omega)$

- There exists an orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ of $L^2(\Omega)$ such that ([Theorem 1, §6.5.1, Evans 2010]) $\phi_n \in H_0^1(\Omega) \cap H^2(\Omega)$ and

$$-\Delta \phi_n = \lambda_n \phi_n,$$

  $\{\lambda_n : n \in \mathbb{N}\}$: a non-decreasing sequence and $\lambda_n \to \infty$ as $n \to \infty$.
- For any $\gamma \in \mathbb{R}$, define

$$\dot{H}^\gamma(\Omega) := \left\{ \sum_{n=0}^\infty c_n \phi_n : \sum_{n=0}^\infty c_n^2 \lambda_n^\gamma < \infty \right\},$$

  inner product : $\left(\sum_{n=0}^\infty c_n \phi_n, \sum_{n=0}^\infty d_n \phi_n \right)_{\dot{H}^\gamma(\Omega)} := \sum_{n=0}^\infty \lambda_n^\gamma c_n d_n$

  for all $\sum_{n=0}^\infty c_n \phi_n, \sum_{n=0}^\infty d_n \phi_n \in \dot{H}^\gamma(\Omega)$, $\|\cdot\|_{\dot{H}^\gamma(\Omega)}$: the norm induced by the inner product.
- $\dot{H}^\gamma(\Omega)$ is a Hilbert space with an orthonormal basis $\{\lambda_n^{-\gamma/2}\phi_n : n \in \mathbb{N}\}$. In addition, $\dot{H}^{-\gamma}(\Omega)$ is the dual space of $\dot{H}^\gamma(\Omega)$ in the sense that

$$\langle \sum_{n=0}^\infty c_n \phi_n, \sum_{n=0}^\infty d_n \phi_n \rangle_{\dot{H}^\gamma(\Omega)} := \sum_{n=0}^\infty c_n d_n$$

  for all $\sum_{n=0}^\infty c_n \phi_n \in \dot{H}^{-\gamma}(\Omega)$ and $\sum_{n=0}^\infty d_n \phi_n \in \dot{H}^\gamma(\Omega)$.

Interpolation spaces

Assume $-\infty < a < b < \infty$.

- For any $m \in \mathbb{N}$, define

$$
\begin{aligned}
{}^0H^m(a,b) &:= \{v \in H^m(a,b) : v^{(k)}(b) = 0,\ 0 \leqslant k < m,\ k \in \mathbb{N}\},\\
{}_0H^m(a,b) &:= \{v \in H^m(a,b) : v^{(k)}(a) = 0,\ 0 \leqslant k < m,\ k \in \mathbb{N}\},
\end{aligned}
$$

  where $v^{(k)}$ is the k-th weak derivative of v, and endow those two spaces with the following norms:

$$
\begin{aligned}
\|v\|_{{}^0H^m(a,b)} &:= \left\|v^{(m)}\right\|_{L^2(a,b)} \quad \forall\, v \in {}^0H^m(a,b),\\
\|v\|_{{}_0H^m(a,b)} &:= \left\|v^{(m)}\right\|_{L^2(a,b)} \quad \forall\, v \in {}_0H^m(a,b).
\end{aligned}
$$

- For $\gamma > 0$, define two interpolation spaces:

$$
\begin{aligned}
{}^0H^\gamma(a,b) &:= [L^2(a,b), {}^0H^m(a,b)]_{\theta,2},\\
{}_0H^\gamma(a,b) &:= [L^2(a,b), {}_0H^m(a,b)]_{\theta,2},
\end{aligned}
$$

  with corresponding interpolation norms defined by (2.1), where $0 < \theta < 1$ and $m \in \mathbb{N}$ such that $\gamma = m\theta$.

Vector-valued space $H^\gamma(a,b;X)$

Now let X be a separable Hilbert space with an inner product $(\cdot,\cdot)_X$ and an orthonormal basis $\{e_i : i \in \mathbb{N}\}$.
For any $\gamma \in \mathbb{R}$, define the vector-valued space

$$H^\gamma(a,b;X) := \left\{ v \in L^2(a,b;X) : \sum_{i=0}^{\infty} \|(v,e_i)_X\|_{H^\gamma(a,b)}^2 < \infty \right\},$$

equipped with the norm

$$\|v\|_{H^\gamma(a,b;X)} := \left(\sum_{i=0}^{\infty} \|(v,e_i)_X\|_{H^\gamma(a,b)}^2 \right)^{1/2} \quad \forall\, v \in H^\gamma(a,b;X).$$

For $\gamma > 0$, the two spaces ${}_0H^\gamma(a,b;X)$ and ${}^0H^\gamma(a,b;X)$ can be defined analogously.

Riemann–Liouville fractional calculus operators

For $\gamma > 0$, define

$$\left(\mathrm{I}_{a+}^{\gamma} v\right)(t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} v(s)\,\mathrm{d}s, \quad t \in (a,b),$$
$$\left(\mathrm{I}_{b-}^{\gamma} v\right)(t) := \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} v(s)\,\mathrm{d}s, \quad t \in (a,b),$$

for all $v \in L^1(a,b;X)$, where $\Gamma(\cdot)$ denotes the Gamma function.
For $j-1 \leqslant \gamma < j$ with $j \in \mathbb{N}_+$, define

$$\mathrm{D}_{a+}^{\gamma} := \mathrm{D}^j\,\mathrm{I}_{a+}^{j-\gamma}, \quad \mathrm{D}_{b-}^{\gamma} := (-1)^j\,\mathrm{D}^j\,\mathrm{I}_{b-}^{j-\gamma},$$

where D is the first-order differential operator in the distribution sense.

Lemma 2.1 (Samko-Kilbas-Marichev 1993)

If $0 < \alpha, \beta < \infty$, *then*

$$\mathrm{I}_{0+}^{\beta}\,\mathrm{I}_{0+}^{\alpha}\,v = \mathrm{I}_{0+}^{\beta+\alpha}\,v, \quad \mathrm{I}_{1-}^{\beta}\,\mathrm{I}_{1-}^{\alpha}\,v = \mathrm{I}_{1-}^{\beta+\alpha}\,v, \quad \forall v \in L^1(0,1);$$

if $0 < \alpha < \beta < \infty$, *then*

$$\mathrm{D}_{0+}^{\beta}\,\mathrm{I}_{0+}^{\alpha}\,v = \mathrm{D}_{0+}^{\beta-\alpha}\,v, \quad \mathrm{D}_{1-}^{\beta}\,\mathrm{I}_{1-}^{\alpha}\,v = \mathrm{D}_{1-}^{\beta-\alpha}\,v, \quad \forall v \in L^1(0,1).$$

Moreover, $\langle \mathrm{I}_{0+}^{\beta}\,v, w\rangle_{(0,1)} = \langle v, \mathrm{I}_{1-}^{\beta}\,w\rangle_{(0,1)}, \quad \forall v, w \in L^2(0,1).$

Lemma 2.2 (Ervin-Roop 2006)

If $0 < \gamma < 1/2$ *and* $v, w \in H^{\gamma}(0,T)$, *then*

$$\begin{aligned}
&\langle \mathrm{D}_{0+}^{\gamma}\,v, \mathrm{D}_{T-}^{\gamma}\,v\rangle_{(0,T)} = \cos\gamma\pi |v|_{H^{\gamma}(0,T)}^2, \\
&\langle \mathrm{D}_{0+}^{\gamma}\,v, \mathrm{D}_{T-}^{\gamma}\,w\rangle_{(0,T)} = \langle \mathrm{D}_{0+}^{2\gamma}\,v, w\rangle_{H^{\gamma}(0,T)} = \langle \mathrm{D}_{T-}^{2\gamma}\,w, v\rangle_{H^{\gamma}(0,T)}, \\
&\cos\gamma\pi \|\mathrm{I}_{0+}^{\gamma}\,v\|_{L^2(0,T)}^2 \leqslant \langle \mathrm{I}_{0+}^{\gamma}\,v, \mathrm{I}_{T-}^{\gamma}\,v\rangle_{(0,T)} \leqslant \sec\gamma\pi \|\mathrm{I}_{0+}^{\gamma}\,v\|_{L^2(0,T)}^2, \\
&\cos\gamma\pi \|\mathrm{D}_{0+}^{\gamma}\,v\|_{L^2(0,T)}^2 \leqslant \langle \mathrm{D}_{0+}^{\gamma}\,v, \mathrm{D}_{T-}^{\gamma}\,v\rangle_{(0,T)} \leqslant \sec\gamma\pi \|\mathrm{D}_{0+}^{\gamma}\,v\|_{L^2(0,T)}^2.
\end{aligned}$$

Lemma 2.3 (Luo-Li-X 2019JSC)

If $v \in {}_0H^\beta(0,1;\dot H^r(\Omega)) \cap {}_0H^\gamma(0,1;\dot H^s(\Omega))$ *with* $\gamma, \beta \geqslant 0$ *and* $s, r \in \mathbb{R}$, *then for all* $0 < \theta < 1$,

$$\|v\|_{{}_0H^{\theta\beta+(1-\theta)\gamma}(0,1;\dot H^{\theta r+(1-\theta)s}(\Omega))} \leqslant C_{\beta,\gamma,\theta}\left(\|v\|_{{}_0H^\beta(0,1;\dot H^r(\Omega))} + \|v\|_{{}_0H^\gamma(0,1;\dot H^s(\Omega))}\right).$$

Similarly, if $v \in {}^0H^\beta(0,1;\dot H^r(\Omega)) \cap {}^0H^\gamma(0,1;\dot H^s(\Omega))$ *with* $\gamma, \beta \geqslant 0$ *and* $s, r \in \mathbb{R}$, *then for all* $0 < \theta < 1$,

$$\|v\|_{{}^0H^{\theta\beta+(1-\theta)\gamma}(0,1;\dot H^{\theta r+(1-\theta)s}(\Omega))} \leqslant C_{\beta,\gamma,\theta}\left(\|v\|_{{}^0H^\beta(0,1;\dot H^r(\Omega))} + \|v\|_{{}^0H^\gamma(0,1;\dot H^s(\Omega))}\right).$$

Lemma 2.4 (Luo-Li-X 2019JSC)

If $\beta \geqslant \gamma > 0$, *then*

$$\begin{aligned} \|\mathrm{D}_{T-}^\gamma v\|_{{}^0H^{\beta-\gamma}(0,T)} &\leqslant C_1\|v\|_{{}^0H^\beta(0,T)} \quad \forall\, v \in {}^0H^\beta(0,T), \\ \|\mathrm{D}_{0+}^\gamma v\|_{{}_0H^{\beta-\gamma}(0,T)} &\leqslant C_2\|v\|_{{}_0H^\beta(0,T)} \quad \forall\, v \in {}_0H^\beta(0,T), \end{aligned}$$

where C_1 *and* C_2 *depend only on* γ *and* β.

Lemma 2.5 (Luo-Li-X 2019JSC)

If $\beta, \gamma \geqslant 0$, *then*

$$C_1 \|v\|_{{}^0H^\beta(0,T)} \leqslant \|\mathrm{I}_{T-}^\gamma \, v\|_{{}^0H^{\beta+\gamma}(0,T)} \leqslant C_2 \|v\|_{{}^0H^\beta(0,T)} \quad \forall\, v \in {}_0H^\beta(0,T),$$

$$C_3 \|v\|_{{}_0H^\beta(0,T)} \leqslant \|\mathrm{I}_{0+}^\gamma \, v\|_{{}_0H^{\beta+\gamma}(0,T)} \leqslant C_4 \|v\|_{{}_0H^\beta(0,T)} \quad \forall\, v \in {}_0H^\beta(0,T).$$

where C_1, C_2, C_3 *and* C_4 *depend only on* γ *and* β.

Lemma 2.6 (Luo-Li-X 2019JSC)

If $0 < \gamma < 1/2$, *then for all* $v \in {}_0H^1(0,1)$,

$$\|v\|_{C[0,1]} \leqslant C_\gamma \|v\|_{{}_0H^1(0,1)}^{(1/2-\gamma)/(1-\gamma)} \|v\|_{{}_0H^\gamma(0,1)}^{1/(2-2\gamma)}.$$

Moreover, if $v \in {}_0H^\gamma(0,1)$ *with* $1/2 < \gamma \leqslant 1$, *then for all* $0 < \epsilon \leqslant 1$,

$$\|v\|_{C[0,1]} \leqslant \frac{C_\gamma}{\sqrt{\epsilon}} \|v\|_{{}_0H^{1/2}(0,1)}^{1-\epsilon} \|v\|_{{}_0H^\gamma(0,1)}^{\epsilon}.$$

Weak solution in the case $u_0 = 0$

Define

$$\mathcal{X} := {}_0H^{\alpha/2}(0,T;L^2(\Omega)) \cap L^2(0,T;\dot{H}^1(\Omega)),$$
$$\mathcal{Y} := {}^0H^{1-\alpha/2}(0,T;L^2(\Omega)) \cap {}^0H^{1-\alpha}(0,T;\dot{H}^1(\Omega)),$$

endowed with the following two norms:

$$\|\cdot\|_{\mathcal{X}} := \left(\|\cdot\|^2_{{}_0H^{\alpha/2}(0,T;L^2(\Omega))} + \|\cdot\|^2_{L^2(0,T;\dot{H}^1(\Omega))} \right)^{1/2},$$
$$\|\cdot\|_{\mathcal{Y}} := \left(\|\cdot\|^2_{{}^0H^{1-\alpha/2}(0,T;L^2(\Omega))} + \|\cdot\|^2_{{}^0H^{1-\alpha}(0,T;\dot{H}^1(\Omega))} \right)^{1/2}.$$

Assuming that $f \in \mathcal{Y}^*$, we call $u \in \mathcal{X}$ a weak solution to problem (1.1) if

$$\langle \mathrm{D}^{\alpha}_{0+} u, v \rangle_{{}_0H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla u, \nabla v \rangle_{\Omega_T} = \langle f, \mathrm{I}^{1-\alpha}_{T-} v \rangle_{\mathcal{Y}} \quad \forall\, v \in \mathcal{X}. \tag{3.1}$$

Since ${}_0H^{\alpha/2}(0,T;L^2(\Omega)) = H^{\alpha/2}(0,T;L^2(\Omega))$ in the sense of equivalent norms and applying Lemma 2.5 implies that

$$\|\mathrm{I}^{1-\alpha}_{T-} v\|_{\mathcal{Y}} \leqslant C_\alpha \|v\|_{\mathcal{X}} \quad \text{for all } v \in \mathcal{X}, \tag{3.2}$$

we readily conclude that the above weak solution is well-defined, according to Lemma 2.2 and the Lax–Milgram theorem.

Theorem 1

If $f \in \mathcal{Y}^*$, then problem (1.1) *admits a unique weak solution $u \in \mathcal{X}$ satisfying $\|u\|_{\mathcal{X}} \leqslant C_\alpha \|f\|_{\mathcal{Y}^*}$.*

Analysis of regularity

We first consider the following problem: seek $y \in {}_0H^{\alpha/2}(0,T)$ such that

$$\langle \mathrm{D}_{0+}^{\alpha}\, y, z\rangle_{{}_0H^{\alpha/2}(0,T)} + \lambda\langle y, z\rangle_{(0,T)} = \langle g, \mathrm{I}_{T-}^{1-\alpha}\, z\rangle_{{}^0H^{1-\alpha/2}(0,T)} \tag{3.3}$$

for all $z \in {}_0H^{\alpha/2}(0,T)$, where $g \in ({}^0H^{1-\alpha/2}(0,T))^*$ and $\lambda > 0$ is a constant.

By Lemmas 2.2 and 2.5 and the Lax–Milgram theorem, we conclude that problem (3.3) admits a unique solution $y \in {}_0H^{\alpha/2}(0,T)$, with

$$\|y\|_{{}_0H^{\alpha/2}(0,T)} + \lambda^{1/2}\|y\|_{L^2(0,T)} \leqslant C_\alpha \|g\|_{({}^0H^{1-\alpha/2}(0,T))^*}.$$

Lemma 3.1

If $g \in L^2(0,T)$, then the solution y to problem (3.3) satisfies

$$\begin{aligned} y' + \lambda\,\mathrm{D}_{0+}^{1-\alpha}\, y &= g, \\ \|y\|_{{}_0H^1(0,T)} + \lambda\|y\|_{{}_0H^{1-\alpha}(0,T)} &\leqslant C_\alpha\|g\|_{L^2(0,T)}. \end{aligned} \tag{3.4}$$

In addition, if $1/2 \leqslant \alpha < 1$, then for all $0 < \epsilon \leqslant 2$,

$$\lambda^{1/(2\alpha)-\sigma\epsilon/2}\|y\|_{C[0,T]} \leqslant \frac{C_{\alpha,T}}{\epsilon^{\sigma/2}}\|g\|_{L^2(0,T)}, \tag{3.5}$$

where $\sigma = 0$ if $1/2 < \alpha < 1$ and $\sigma = 1$ if $\alpha = 1/2$.

Lemma 3.2
If $g \in {}_0H^\gamma(0,T)$ *with* $0 < \gamma \leqslant 1/2$*, then the solution* y *to problem* (3.3) *satisfies that*

$$\|y\|_{{}_0H^{1+\gamma}(0,T)} + \lambda \|y\|_{{}_0H^{1+\gamma-\alpha}(0,T)} \leqslant C_{\alpha,\gamma} \|g\|_{{}_0H^\gamma(0,T)}. \tag{3.6}$$

Weak solution u to problem (1.1)

If $f \in L^2(0,T;\dot{H}^{-1}(\Omega))$, then the weak solution u to problem (1.1) is ([Theorem 3.1, Li-Luo-X 2019 SINUM])

$$u(t) = \sum_{n=0}^{\infty} y_n(t)\phi_n, \quad 0 < t \leqslant T,$$

where $y_n \in {}_0H^{\alpha/2}(0,T)$ satisfies that

$$\langle \mathrm{D}_{0+}^\alpha\, y_n, z\rangle_{{}_0H^{\alpha/2}(0,T)} + \lambda_n \langle y_n, z\rangle_{(0,T)} = \langle \langle f, \phi_n\rangle_{\dot{H}^1(\Omega)}, \mathrm{I}_{T-}^{1-\alpha}\, z\rangle_{(0,T)}$$

for all $z \in {}_0H^{\alpha/2}(0,T)$.

Regularity of weak solution to problem (1.1)*:* $u_0 = 0$

Therefore, the desired regularity results follow from Lemmas 3.1 and 3.2.

Theorem 3.1

Assume that $f \in {}_0H^{\gamma}(0,T;\dot H^{-\beta}(\Omega))$ *with* $0 \leqslant \gamma \leqslant 1/2$ *and* $0 \leqslant \beta \leqslant 1$*. Then the weak solution* u *to problem* (1.1) *satisfies*

$$u' - \mathrm{D}_{0+}^{1-\alpha}\,\Delta u = f \quad \text{in } L^2(0,T;\dot H^{-\beta}(\Omega)),$$

$$\|u\|_{{}_0H^{1+\gamma}(0,T;\dot H^{-\beta}(\Omega))} + \|u\|_{{}_0H^{1+\gamma-\alpha}(0,T;\dot H^{2-\beta}(\Omega))} \leqslant C_{\alpha,\gamma}\|f\|_{{}_0H^{\gamma}(0,T;\dot H^{-\beta}(\Omega))}.$$

In addition, if $\gamma = 0$ *and* $1/2 \leqslant \alpha < 1$*, then for all* $0 < \epsilon \leqslant 2$*,*

$$\|u\|_{C([0,T];\dot H^{1/\alpha-\sigma\epsilon-\beta}(\Omega))} \leqslant \frac{C_{\alpha,T}}{\epsilon^{\sigma/2}}\|f\|_{L^2(0,T;\dot H^{-\beta}(\Omega))},$$

where $\sigma = 0$ *if* $1/2 < \alpha < 1$ *and* $\sigma = 1$ *if* $\alpha = 1/2$*.*

Regularity of weak solution to dual problem

For the dual problem of (1.1), we have the following theorem.

Theorem 3.2

Assume that $q \in L^2(0,T;\dot H^{-\beta}(\Omega))$ *with* $0 \leqslant \beta \leqslant 1$. *Then there exists a unique*

$$w \in \mathcal{G} := {}^0H^1(0,T;\dot H^{-\beta}(\Omega)) \cap {}^0H^{1-\alpha}(0,T;\dot H^{2-\beta}(\Omega))$$

such that

$$-w' - \mathrm{D}_{T-}^{1-\alpha}\,\Delta w = q$$

and

$$\|w\|_{{}^0H^1(0,T;\dot H^{-\beta}(\Omega))} + \|w\|_{{}^0H^{1-\alpha}(0,T;\dot H^{2-\beta}(\Omega))} \leqslant C_\alpha \|q\|_{L^2(0,T;\dot H^{-\beta}(\Omega))}.$$

Weak solution in the case $f = 0$

For $a, b > 0$, recall the Mittag-Leffler function

$$E_{a,b}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}, \quad z \in \mathbb{C}.$$

Given $\lambda, t > 0$ and $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$, we have (cf. [Gorenflo et. al 2014]):

$$|E_{a,b}(-t)| \leqslant \frac{C_{a,b}}{1+t}, \tag{3.7}$$

$$\mathrm{D}_{0+}^{\gamma} E_{a,1}(-\lambda t^a) = t^{-\gamma} E_{a,1-\gamma}(-\lambda t^a), \tag{3.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{a,1}(-\lambda t^a) = -\lambda t^{a-1} E_{a,a}(-\lambda t^a). \tag{3.9}$$

For any $\lambda > 0$ and $y_0 \in \mathbb{R}$, by (3.8) and (3.9), it is easy to see that

$$y(t) = y_0 E_{\alpha,1}(-\lambda t^{\alpha}), \quad 0 \leqslant t \leqslant T$$

solves the equation

$$y' + \lambda \, \mathrm{D}_{0+}^{1-\alpha} y = 0, \quad 0 < t \leqslant T,$$

with initial condition $y(0) = y_0$.

Formulation of weak solution using Mittag-Leffler function

Therefore, for $f = 0$ and $u_0 \in \dot{H}^{-2}(\Omega)$, it is natural to define a weak solution of problem (1.1) by that [Sakamoto-Yamamoto 2011]

$$u(t) := \sum_{n=0}^{\infty} E_{\alpha,1}(-\lambda_n t^{\alpha})\langle u_0, \phi_n\rangle_{\dot{H}^2(\Omega)}\phi_n, \quad 0 \leqslant t \leqslant T. \tag{3.10}$$

It follows from (3.7) that $u \in C([0,T]; \dot{H}^{-2}(\Omega))$ is well defined. In addition, we have $u(0) = u_0$ and

$$\|u\|_{C([0,T];\dot{H}^{-2}(\Omega))} \leqslant C_{\alpha}\|u_0\|_{\dot{H}^{-2}(\Omega)}.$$

Since $u_0 \in \dot{H}^{-2}(\Omega)$, (3.10) shall be understood as the "very weak solution" by using the transposition method [Lions 1972].

Regularity of weak solution with $u_0 \in L^2(\Omega)$

Theorem 3.3

If $u_0 \in L^2(\Omega)$, then the weak solution defined by (3.10) satisfies

$$\langle u', v \rangle_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} + \langle \mathrm{D}_{0+}^{1-\alpha} \nabla u, \nabla v \rangle_{H^{(1-\alpha)/2}(0,T;L^2(\Omega))} = 0, \quad (3.11)$$

for all $v \in H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))$, and there holds

$$\begin{aligned} &\|u'\|_{(H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega)))^*} + \|u\|_{C([0,T];L^2(\Omega))} + \|u\|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} \\ &\qquad + \epsilon_{\alpha,\gamma}\|u\|_{L^2(0,T;\dot{H}^\gamma(\Omega))} \leqslant C_{\alpha,T}\|u_0\|_{L^2(\Omega)}, \end{aligned} \quad (3.12)$$

where $\epsilon_{\alpha,\gamma} := \sqrt{2-\gamma} + \sqrt{|2\alpha - 1|}$ with $\gamma = \min\{2, 1/\alpha\}$ if $\alpha \neq 1/2$ and $1 \leqslant \gamma < 2$ if $\alpha = 1/2$.

Finite element spaces

- Temporal partition: Given $J \in \mathbb{N}_{>0}$, let $0 = t_0 < t_1 < \cdots < t_J = T$ be a partition of $[0, T]$ with $\tau := \max_{1 \leqslant j \leqslant J}(t_j - t_{j-1})$, and set $I_j := (t_{j-1}, t_j)$ for $1 \leqslant j \leqslant J$.
- Spatial partition: Let $\mathcal{K}_h$ be a conventional conforming and quasi-uniform triangulation of Ω consisting of d-simplexes, and use h to denote the maximum diameter of the elements in $\mathcal{K}_h$.

Define finite element spaces

$$S_h := \left\{ v_h \in H_0^1(\Omega) : \ v_h|_K \in P_1(K) \quad \forall\, K \in \mathcal{K}_h \right\},$$

$$\mathcal{X}_{\tau,h} := \left\{ V \in L^2(0, T; S_h) : \ V|_{I_j} \in P_0(I_j; S_h) \quad \forall\, 1 \leqslant j \leqslant J \right\},$$

where $P_1(K)$ is the set of all linear polynomials defined on K, and $P_0(I_j; S_h)$ is the set of all S_h-valued constant functions on I_j.

For each $V \in \mathcal{X}_{\tau,h}$, set

$$V_j^+ := \lim_{t \to t_j+} V(t) \quad \text{for } 0 \leqslant j < J, \text{ and } V_J^+ := 0;$$

$$V_j^- := \lim_{t \to t_j-} V(t) \quad \text{for } 1 \leqslant j \leqslant J, \text{ and } V_0^- := 0;$$

$$[\![V_j]\!] := V_j^+ - V_j^- \quad \text{for } 0 \leqslant j \leqslant J.$$

Piecewise constant DG method

Assume that $u_0 \in S_h^*$ and $f \in \mathcal{X}_{\tau,h}^*$. Find $U \in \mathcal{X}_{\tau,h}$ such that (cf. [McLean-Mustapha 2009])

$$\mathcal{A}(U, V) = \langle f, V \rangle_{\mathcal{X}_{\tau,h}} + \langle u_0, V_0^+ \rangle_{S_h} \quad \forall\, V \in \mathcal{X}_{\tau,h}, \tag{4.1}$$

where

$$\mathcal{A}(W, V) := \sum_{j=0}^{J-1} \langle [\![W_j]\!], V_j^+ \rangle_\Omega + \langle \mathrm{D}_{0+}^{1-\alpha} \nabla W, \nabla V \rangle_{\Omega_T}$$

for all $W, V \in \mathcal{X}_{\tau,h}$.

From [Theorem 12.1, Thomee 2006] and Lemma 2.2 it follows

$$\begin{aligned} \mathcal{A}(V, V\chi_{(0,t_j)}) \geqslant \frac{1}{2} \left(\|V_j^-\|_{L^2(\Omega)}^2 + \|V_0^+\|_{L^2(\Omega)}^2 \right) \\ + \sin \frac{\alpha\pi}{2} |V|_{H^{(1-\alpha)/2}(0,t_j;\dot{H}^1(\Omega))}^2, \end{aligned} \tag{4.2}$$

for all $V \in \mathcal{X}_{\tau,h}$ and $1 \leqslant j \leqslant J$, where $\chi_{(a,b)}$ denotes the indicator function of the interval (a, b).

Well-posedness of the scheme

- For convenience, in what follows we assume that u is the weak solution to problem (1.1) and U is its numerical approximation defined by (4.1).
- The notation $a \lesssim b$ means that there exists a generic positive constant C, independent of h, τ and u, such that $a \leqslant Cb$. Moreover, $a \sim b$ means $a \lesssim b \lesssim a$.

Theorem 4.1

If $u_0 \in L^2(\Omega)$ and $f \in (H^{(1-\alpha)/2}(0,T;\dot H^1(\Omega)))^*$, then problem (4.1) *admits a unique solution U such that*

$$\begin{aligned} &\|U\|_{L^\infty(0,T;L^2(\Omega))} + |U|_{H^{(1-\alpha)/2}(0,T;\dot H^1(\Omega))} \\ \lesssim\ &\|u_0\|_{L^2(\Omega)} + \|f\|_{(H^{(1-\alpha)/2}(0,T;\dot H^1(\Omega)))^*}. \end{aligned} \tag{4.3}$$

Main error estimates

Theorem 4.2
If $u_0 = 0$ and $f \in L^2(\Omega_T)$, then

$$\|u - U\|_{L^2(\Omega_T)} \lesssim (\tau + h^2)\|f\|_{L^2(\Omega_T)}, \tag{4.4}$$

$$|u - U|_{H^{(1-\alpha)/2}(0,T;\dot{H}^1(\Omega))} \lesssim (\tau^{1/2} + h)\|f\|_{L^2(\Omega_T)}. \tag{4.5}$$

Theorem 4.3
Assume that $f = 0$. If $u_0 \in L^2(\Omega)$, then

$$\|u - U\|_{L^2(\Omega_T)} \lesssim (\tau^{1/2} + h)\|u_0\|_{L^2(\Omega)}. \tag{4.6}$$

Remark 4.1
In view of Lemma 2.3, Theorems 3.1 and 3.3, we conclude that

- *(4.4) and (4.5) are optimal with respect to the solution regularity*
- *(4.6) only gives optimal temporal accuracy $O(\tau^{1/2})$, since the optimal spatial accuracy should be $\min\{2, 1/\alpha\}$ (cf. (3.12)).*
- *It is possible to recover the first order accuracy $O(J^{-1})$ by using graded temporal meshes (cf. numerical results in Table 5.7).*

Quasi-optimal estimates inr $L^\infty(L^2)$-norm: uniform temporal grid

Moreover, if the temporal grid is equi-distributed, then quasi-optimal (including logarithm factors) error bounds under the $L^\infty(0,T;L^2(\Omega))$-norm are derived.

Theorem 4.4
Assume $u_0 = 0$ and the temporal grid is uniform. If $f \in L^2(\Omega_T)$, then

$$\|u - U\|_{L^\infty(0,T;L^2(\Omega))} \lesssim |\ln \tau| \Big(\tau^{1/2} + \epsilon_h h^{\min\{2,1/\alpha\}} \Big) \|f\|_{L^2(\Omega_T)}, \qquad (4.7)$$

where $\epsilon_h = 1$ if $\alpha \neq 1/2$ and $\epsilon_h = \sqrt{|\ln h|}$ if $\alpha = 1/2$. Moreover, if $f \in {}_0H^{1/2}(0,T;L^2(\Omega))$, then

$$\|u - U\|_{L^\infty(0,T;L^2(\Omega))} \lesssim |\ln \tau| \left(|\ln \tau| \tau + h^2 \right) \|f\|_{{}_0H^{1/2}(0,T;L^2(\Omega))}. \qquad (4.8)$$

In this section, we present several numerical experiments to verify the theoretical results with $T = 1$ and $\Omega = (0, 1)$.

- We use uniform spatial grids, and introduce the following notations:

$$\begin{aligned} \mathcal{E}_1 &:= \|\widehat{u} - U\|_{L^2(\Omega_T)}, \\ \mathcal{E}_2 &:= \|\widehat{u} - U\|_{L^\infty(0,T;L^2(\Omega))}, \\ \mathcal{E}_3 &:= \sqrt{\langle \mathrm{D}_{0+}^{1-\alpha}(\nabla\widehat{u} - \nabla U), \nabla\widehat{u} - \nabla U\rangle_{\Omega_T}}, \end{aligned}$$

  where the reference solution $\widehat{u}$ is the numerical solution with respect to $h = 2^{-10}$ and $\tau = 2^{-15}$ under uniform grids both in time and space.

- Note that, by Lemma 2.2,

$$\mathcal{E}_3 \sim \|\mathrm{D}_{0+}^{(1-\alpha)/2}(\nabla\widehat{u} - \nabla U)\|_{L^2(\Omega_T)} \sim \|\widehat{u} - U\|_{H^{\frac{1-\alpha}{2}}(0,T;\dot{H}^1(\Omega))}.$$

- With uniform temporal grids, the DG scheme (4.1) leads to a block triangular Toeplitz-like with tri-diagonal block system, and we can adopt the fast direct method proposed in [Ke-Ng-Sun 2015] to solve it efficiently with quasi-optimal complexity $O((\tau h)^{-1}|\ln\tau|^2)$. Moreover, $\mathcal{E}_3$ can be computed via fast Fourier transform.

Two experiments with $u_0(x) = 0$: uniform grids

Experiment 1. Consider

$$\begin{aligned} u_0(x) &:= 0, & x &\in \Omega, \\ f(x,t) &:= x^{-0.49} t^{-0.49}, & (x,t) &\in \Omega_T. \end{aligned}$$

To test the accuracy of spatial discretization, we fix temporal step size $\tau = 2^{-15}$. Since $f \in L^2(\Omega_T)$, according to Theorems 4.2 and 4.4, we have

$$\mathcal{E}_1 = O(h^2), \quad \mathcal{E}_2 = O\big(h^{\min\{2,1/\alpha\}}\big), \quad \mathcal{E}_3 = O(h).$$

These coincide with the numerical results in Table 5.1.

	h	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order	$\mathcal{E}_3$	Order
$\alpha = 0.8$	2^{-2}	2.00e-02	–	3.67e-02	–	2.87e-01	–
	2^{-3}	5.53e-03	1.85	1.48e-02	1.31	1.59e-01	0.85
	2^{-4}	1.50e-03	1.88	5.95e-03	1.31	8.68e-02	0.87
	2^{-5}	4.04e-04	1.90	2.42e-03	1.30	4.67e-02	0.89
$\alpha = 0.2$	2^{-4}	1.13e-03	–	1.45e-03	–	6.05e-02	–
	2^{-5}	3.02e-04	1.90	3.88e-04	1.91	3.21e-02	0.91
	2^{-6}	7.99e-05	1.92	1.02e-04	1.92	1.69e-02	0.93
	2^{-7}	2.08e-05	1.94	2.67e-05	1.94	8.81e-03	0.94

Table 5.1: Spatial errors of **Experiment 1** with $\tau = 2^{-15}$.

Next, we consider temporal errors and choose $h = 2^{-10}$.
By Theorems 4.2 and 4.4, we have

$$\mathcal{E}_1 = O(\tau), \quad \mathcal{E}_2 = O(\tau^{1/2}), \quad \mathcal{E}_3 = O(\tau^{1/2}).$$

They match well with the numerical results.

	τ	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order	$\mathcal{E}_3$	Order
$\alpha = 0.7$	2^{-9}	1.99e-03	–	9.22e-02	–	2.12e-02	–
	2^{-10}	1.13e-03	0.81	6.54e-02	0.49	1.42e-02	0.58
	2^{-11}	6.24e-04	0.86	4.43e-02	0.56	9.28e-03	0.61
	2^{-12}	3.27e-04	0.93	2.81e-02	0.66	5.86e-03	0.66
$\alpha = 0.3$	2^{-9}	6.24e-04	–	3.37e-02	–	2.92e-02	–
	2^{-10}	3.63e-04	0.78	2.49e-02	0.43	2.10e-02	0.48
	2^{-11}	2.06e-04	0.82	1.77e-02	0.50	1.48e-02	0.51
	2^{-12}	1.12e-04	0.88	1.17e-02	0.59	1.00e-02	0.56

Table 5.2: Temporal errors of **Experiment 1** with $h = 2^{-10}$

Experiment 2. Consider

$$
\begin{aligned}
u_0(x) &:= 0, && x \in \Omega, \\
f(x,t) &:= x^{-0.49} t^{0.01}, && (x,t) \in \Omega_T.
\end{aligned}
$$

It is clear that $f \in {}_0H^{1/2}(0,T;L^2(\Omega))$. In Tables 5.3 and 5.4, we observe the optimal convergence order $\mathcal{E}_2 = O(\tau + h^2)$, which agrees with Theorem 4.4.

	$\alpha = 0.9$		$\alpha = 0.5$		$\alpha = 0.3$	
h	$\mathcal{E}_2$	Order	$\mathcal{E}_2$	Order	$\mathcal{E}_2$	Order
2^{-4}	7.10e-04	–	5.81e-04	–	5.18e-04	–
2^{-5}	1.90e-04	1.91	1.55e-04	1.90	1.39e-04	1.90
2^{-6}	5.01e-05	1.92	4.11e-05	1.92	3.66e-05	1.92
2^{-7}	1.30e-05	1.94	1.07e-05	1.94	9.55e-06	1.94

Table 5.3: Spatial errors of **Experiment 2** with $\tau = 2^{-15}$.

	$\alpha = 0.7$		$\alpha = 0.4$		$\alpha = 0.1$	
τ	$\mathcal{E}_2$	Order	$\mathcal{E}_2$	Order	$\mathcal{E}_2$	Order
2^{-8}	3.18e-04	–	2.02e-04	–	2.14e-04	–
2^{-9}	1.60e-04	1.00	1.00e-04	1.01	1.04e-04	1.04
2^{-10}	7.95e-05	1.01	4.97e-05	1.01	5.03e-05	1.04
2^{-11}	3.92e-05	1.02	2.46e-05	1.02	2.43e-05	1.05

Table 5.4: Temporal errors of **Experiment 2** with $h = 2^{-10}$.

An experiment with $f = 0$*: uniform grids*

Experiment 3. In this test, let us verify Theorem 4.3 and take

$$u_0(x) := x^{-0.49}, \quad x \in \Omega,$$
$$f(x,t) := 0, \qquad (x,t) \in \Omega_T.$$

The optimal temporal convergence rate $\mathcal{E}_1 = O(\tau^{1/2})$ in Table 5.5 coincides with Theorem 4.3.

	$\alpha = 0.9$		$\alpha = 0.6$		$\alpha = 0.3$	
τ	$\mathcal{E}_1$	Order	$\mathcal{E}_1$	Order	$\mathcal{E}_1$	Order
2^{-7}	2.90e-02	–	2.18e-02	–	1.09e-02	–
2^{-8}	2.00e-02	0.54	1.46e-02	0.58	8.07e-03	0.44
2^{-9}	1.37e-02	0.54	9.77e-03	0.58	5.82e-03	0.47
2^{-10}	9.36e-03	0.55	6.53e-03	0.58	4.07e-03	0.52

Table 5.5: Temporal errors of **Experiment 3** with $h = 2^{-10}$.

However, as mentioned in Remark 4.1,

- Theorem 4.3 only gives suboptimal spatial rate $\mathcal{E}_1 = O(h)$;
- The optimal order should be $\mathcal{E}_1 = O(h^{\min\{2,1/\alpha\}})$, which can be observed from Table 5.6.

	$\alpha = 0.8$		$\alpha = 0.5$		$\alpha = 0.2$	
h	$\mathcal{E}_1$	Order	$\mathcal{E}_1$	Order	$\mathcal{E}_1$	Order
2^{-2}	3.37e-02	–	1.54e-02	–	1.10e-02	–
2^{-3}	1.36e-02	1.31	4.49e-03	1.78	3.03e-03	1.86
2^{-4}	5.31e-03	1.36	1.27e-03	1.82	8.20e-04	1.89
2^{-5}	1.90e-03	1.48	3.48e-04	1.86	2.19e-04	1.90

Table 5.6: Spatial errors of **Experiment 3** with $\tau = 2^{-15}$.

Three cases: graded temporal grids

The rate $O(\tau^{1/2})$ in Theorem 4.2, optimal with respect to the Sobolev regularity, can be further improved via graded grids, provided that the solution possesses some growth estimates like (1.2).

Experiment 4.
Let us investigate the performance of the DG scheme (4.1) under graded temporal grid $t_j = (j/J)^\sigma$, $j = 0, 1, \cdots, J$, with $\sigma > 1$.
We only compute the quantity $\mathcal{E}_2$, which corresponds to the $L^\infty(0, T; L^2(\Omega))$-norm, and consider three cases:

- *Case 1*: $u_0(x) = x^{-0.49}$, $f(x,t) = 0$;
- *Case 2*: $u_0(x) = 0$, $f(x,t) = x^{-0.49}t^{-0.49}$;
- *Case 3*: $u_0(x) = 0$, $f(x,t) = x^{-0.49}|1-2t|^{-0.49}$.

Note that for all cases we have $u_0 \in L^2(\Omega)$ and $f \in L^2(\Omega_T)$.

- According to [McLean 2010], one can obtain growth estimates for the first two cases, and the first order accuracy $\mathcal{E}_2 = O(J^{-1})$ is maintained with suitable parameter $\sigma > 1$; see Tables 5.7 and 5.8.

Improve accuracy with suitable σ: Case 1

Case 1: $u_0(x) = x^{-0.49}$, $f(x,t) = 0$

$\alpha = 0.3$				$\alpha = 0.9$			
σ	J	$\mathcal{E}_2$	Order	σ	J	$\mathcal{E}_2$	Order
2	2^5	9.91e-01	–	1.5	2^5	1.17e-00	–
	2^6	8.59e-01	0.21		2^6	9.80e-01	0.26
	2^7	7.26e-01	0.24		2^7	7.77e-01	0.33
	2^8	5.98e-01	0.28		2^8	5.68e-01	0.45
5	2^5	1.09e-00	–	2.5	2^5	5.17e-01	–
	2^6	8.73e-01	0.32		2^6	3.09e-01	0.74
	2^7	6.42e-01	0.44		2^7	1.67e-01	0.89
	2^8	4.16e-01	0.63		2^8	8.62e-02	0.95
9	2^5	3.81e-01	–	4	2^5	2.38e-01	–
	2^6	2.03e-01	0.91		2^6	1.25e-01	0.93
	2^7	1.02e-01	0.99		2^7	6.24e-02	1.00
	2^8	4.98e-02	1.03		2^8	3.09e-02	1.00

Table 5.7: Temporal accuracy of *Case 1* in **Experiment 4**.

Improve accuracy with suitable σ: Case 2

Case 2: $u_0(x) = 0,\ f(x,t) = x^{-0.49}t^{-0.49}$

		$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.8$	
σ	J	$\mathcal{E}_2$	Order	$\mathcal{E}_2$	Order	$\mathcal{E}_2$	Order
1.5	2^5	3.94e-02	–	7.99e-02	–	1.75e-01	–
	2^6	2.67e-02	0.56	5.81e-02	0.46	1.16e-01	0.60
	2^7	1.78e-02	0.58	4.02e-02	0.53	7.33e-02	0.66
	2^8	1.16e-02	0.62	2.64e-02	0.61	4.49e-02	0.71
2.5	2^5	1.16e-02	–	2.56e-02	–	4.90e-02	–
	2^6	5.93e-03	0.97	1.31e-02	0.97	2.46e-02	0.99
	2^7	2.94e-03	1.01	6.62e-03	0.98	1.22e-02	1.01
	2^8	1.46e-03	1.01	3.26e-03	1.02	6.01e-03	1.02

Table 5.8: Temporal accuracy of *Case 2* in **Experiment 4**.

Fail to improve accuracy: Case 3

Case 3: $u_0(x) = 0,\ f(x,t) = x^{-0.49}|1-2t|^{-0.49}$

In this case, it seems hard (or even impossible) to obtain growth estimate of the solution, and the accuracy $\mathcal{E}_2 = O(\tau^{1/2})$ can not be improved; see Table 5.9.

		$\sigma = 1.5$		$\sigma = 2.5$		$\sigma = 5$		$\sigma = 10$	
α	J	$\mathcal{E}_2$	Order	$\mathcal{E}_2$	Order	$\mathcal{E}_2$	Order	$\mathcal{E}_2$	Order
0.1	2^8	2.44e-02	–	3.03e-02	–	4.06e-02	–	5.45e-02	–
	2^9	1.80e-02	0.43	2.24e-02	0.43	3.01e-02	0.43	4.04e-02	0.43
	2^{10}	1.32e-02	0.45	1.64e-02	0.45	2.21e-02	0.45	2.96e-02	0.45
	2^{11}	9.56e-03	0.47	1.19e-02	0.47	1.60e-02	0.47	2.14e-02	0.47
0.2	2^9	2.58e-02	–	3.08e-02	–	3.92e-02	–	4.98e-02	–
	2^{10}	1.99e-02	0.37	2.39e-02	0.37	3.04e-02	0.37	3.87e-02	0.36
	2^{11}	1.51e-02	0.40	1.82e-02	0.39	2.32e-02	0.39	2.96e-02	0.39
	2^{12}	1.10e-02	0.46	1.33e-02	0.45	1.70e-02	0.45	2.18e-02	0.44

Table 5.9: Temporal accuracy of *Case 3* in **Experiment 4**.

Summary

For time fractional subdiffusion problems

- Regularity results for weak solutions are established by using variational approach and Mittag-Leffler function
- Error estimates are derived for the piecewise constant DG method, with low regularity data
- Numerical experiments are conducted to verify the theoretical results

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THANK YOU !