A fast finite volume method for spatial fractional diffusion equations on nonuniform meshes

Zhi-Wei FANG
Foshan University

Joint work with Dr. **Jia-Li ZHANG** and Prof. **Hai-Wei SUN**

6th CSRC Conference
Numerical Methods for Fractional-Derivative Problems

Outline

$$u_t - \mathcal{D}\eta(x)\Big(\gamma\ {}_a\mathcal{D}_x^{1-\alpha}u - (1-\gamma)\ {}_x\mathcal{D}_b^{1-\alpha}u\Big) = f(x,t),$$
$$(x,t) \in (a,b) \times (0,T],$$

with the initial condition

$$u(x,0) = \phi(x),\ \ x \in [a,b],$$

and homogeneous Dirichlet boundary conditions

$$u(a,t) = u(b,t) = 0,\ \ t \in (0,T],$$

$$u_t - \mathcal{D}\eta(x)\Big(\gamma\ {}_a\mathcal{D}_x^{1-\alpha} u - (1-\gamma)\ {}_x\mathcal{D}_b^{1-\alpha} u\Big) = f(x,t),$$
$$(x,t) \in (a,b) \times (0,T],$$

with the initial condition

$$u(x,0) = \phi(x),\ \ x \in [a,b],$$

and homogeneous Dirichlet boundary conditions

$$u(a,t) = u(b,t) = 0,\ \ t \in (0,T],$$

$$u_t - \mathcal{D}\eta(x)\Big(\gamma\ {}_a\mathcal{D}_x^{1-\alpha} u - (1-\gamma)\ {}_x\mathcal{D}_b^{1-\alpha} u\Big) = f(x,t),$$
$$(x,t) \in (a,b) \times (0,T],$$

with the initial condition

$$u(x,0) = \phi(x),\ \ x \in [a,b],$$

and homogeneous Dirichlet boundary conditions

$$u(a,t) = u(b,t) = 0,\ \ t \in (0,T],$$

$$u_t - \mathcal{D}\eta(x)\Big(\gamma\ {}_a\mathcal{D}_x^{1-\alpha} u - (1-\gamma)\ {}_x\mathcal{D}_b^{1-\alpha} u\Big) = f(x,t),$$
$$(x,t) \in (a,b) \times (0,T],$$

$$ {}_a\mathcal{I}_x^{\alpha} w(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{w(\xi)}{(x-\xi)^{1-\alpha}} \mathrm{d}\xi,$$
$$ {}_x\mathcal{I}_b^{\alpha} w(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{w(\xi)}{(\xi-x)^{1-\alpha}} \mathrm{d}\xi,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

$$ {}_a\mathcal{D}_x^{1-\alpha} := \mathcal{D}({}_a\mathcal{I}_x^{\alpha}), \quad {}_x\mathcal{D}_b^{1-\alpha} := -\mathcal{D}({}_x\mathcal{I}_b^{\alpha}).$$

Uniform grids vs Non-uniform grids

Uniform grids
Toeplitz-like structure
Matrix-vector multiplication: $\mathcal{O}(m \log m)$ complixity

Non-uniform grids

Uniform grids vs Non-uniform grids

Uniform grids

Toeplitz-like structure

Matrix-vector multiplication: $\mathcal{O}(m \log m)$ complixity

Non-uniform grids

Dense coefficient matrix without highlighted structure

Matrix-vector multiplication: $\mathcal{O}(m^2)$ complixity

Related Works

Non-uniform meshes

Zhao L. & Deng W. (2016). Adv. Comput. Math.

Simmons, A., Yang, Q., & Moroney, T. (2017). J. Comput. Phys.

Locally refined composite mesh - Toeplitz block

Jia, J. & Wang, H. (2015). J. Comput. Phys.

Jia, J. & Wang, H. (2019). Comput. Math. Appl.

Dai, P., Jia, J., Wang, H., Wu, Q., & Zheng, X. (2021). NLAA

Outline

Discretization

For nonuniform grids $\{x_i\}_{i=0}^{m}$,

Denote $x_{i-\frac{1}{2}} = \frac{x_{i-1}+x_i}{2}$ and ${}_{i-1}\delta_i = x_i - x_{i-1}$,

$\delta_{\min} = \min\limits_{1\le i\le m} {}_{i-1}\delta_i$ and $\delta_{\max} = \max\limits_{1\le i\le m} {}_{i-1}\delta_i$.

Control volume: $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_t(x,t)\mathrm{d}x - \eta(x)\Big(\gamma\ {}_a\mathcal{D}_x^{1-\alpha}u(x,t) - (1-\gamma)\ {}_x\mathcal{D}_b^{1-\alpha}u(x,t)\Big)\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$$
$$= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x,t)\mathrm{d}x.$$

Discretization

For nonuniform grids $\{x_i\}_{i=0}^{m}$,

Denote $x_{i-\frac{1}{2}} = \frac{x_{i-1}+x_i}{2}$ and ${}_{i-1}\delta_i = x_i - x_{i-1}$,

$\delta_{\min} = \min\limits_{1\le i\le m} {}_{i-1}\delta_i$ and $\delta_{\max} = \max\limits_{1\le i\le m} {}_{i-1}\delta_i$.

Control volume: $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

$$
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_t(x,t)\mathrm{d}x - \eta(x)\Big(\gamma\ {}_a\mathcal{D}_x^{1-\alpha}u(x,t) - (1-\gamma)\ {}_x\mathcal{D}_b^{1-\alpha}u(x,t)\Big)\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}
= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x,t)\mathrm{d}x.
$$

Discretization

For nonuniform grids $\{x_i\}_{i=0}^{m}$,

Denote $x_{i-\frac{1}{2}} = \frac{x_{i-1}+x_i}{2}$ and ${}_{i-1}\delta_i = x_i - x_{i-1}$,

$\delta_{\min} = \min\limits_{1\le i\le m} {}_{i-1}\delta_i$ and $\delta_{\max} = \max\limits_{1\le i\le m} {}_{i-1}\delta_i$.

Control volume: $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_t(x,t)\mathrm{d}x - \eta(x)\Big(\gamma\ {}_a\mathcal{D}_x^{1-\alpha} u(x,t) - (1-\gamma)\ {}_x\mathcal{D}_b^{1-\alpha} u(x,t)\Big)\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$$
$$= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x,t)\mathrm{d}x.$$

Discretization

$$
\begin{aligned}
{}_a\mathcal{D}_x^{1-\alpha} u(x,t)\Big|_{x_{s-\frac{1}{2}}} &= \mathcal{D}\,{}_a\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_{s-\frac{1}{2}}} \\
&\approx \frac{1}{{}_{s-1}\delta_s}\left[{}_a\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_s} - {}_a\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_{s-1}} \right].
\end{aligned}
$$

Using piecewise constant approximation of $u(x,t)$ leads to dense matrix without Toeplitz-like structure.

$\rightarrow$ $\mathcal{O}(m^2)$ Storage & $\mathcal{O}(m^3)$ Operations

Discretization

$$
\begin{aligned}
{}_a\mathcal{D}_x^{1-\alpha} u(x,t)\Big|_{x_{s-\frac{1}{2}}} &= \mathcal{D}\, {}_a\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_{s-\frac{1}{2}}} \\
&\approx \frac{1}{{}_{s-1}\delta_s}\left[{}_a\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_s} - {}_a\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_{s-1}} \right].
\end{aligned}
$$

Using piecewise constant approximation of $u(x,t)$ leads to dense matrix without Toeplitz-like structure.

$\rightarrow$ $\mathcal{O}(m^2)$ Storage & $\mathcal{O}(m^3)$ Operations

SOE technique

$$
\begin{aligned}
{}_a\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_s} &\approx \frac{1}{\Gamma(\alpha)}\left(\int_a^{x_{s-1}} \frac{\bar{u}(\xi,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi + \int_{x_{s-1}}^{x_s} \frac{\bar{u}(\xi,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi\right)\\
&\equiv \mathcal{I}_\delta^{a,s}(t) + \mathcal{I}_\delta^{s-1,s}(t),
\end{aligned}
$$

where

$$
\mathcal{I}_\delta^{s-1,s}(t) = \frac{1}{\Gamma(\alpha)}\int_{x_{s-1}}^{x_s} \frac{u(x_s,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi = \frac{({}_{s-1}\delta_s)^{\alpha} u(x_s,t)}{\Gamma(1+\alpha)}.
$$

SOE technique

$$
{}_a\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_s} \approx \frac{1}{\Gamma(\alpha)}\left(\int_a^{x_{s-1}} \frac{\bar{u}(\xi,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi + \int_{x_{s-1}}^{x_s} \frac{\bar{u}(\xi,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi\right)
\equiv \mathcal{I}_{\delta}^{a,s}(t) + \mathcal{I}_{\delta}^{s-1,s}(t),
$$

where

$$
\mathcal{I}_{\delta}^{s-1,s}(t) = \frac{1}{\Gamma(\alpha)}\int_{x_{s-1}}^{x_s} \frac{u(x_s,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi = \frac{({}_{s-1}\delta_s)^{\alpha} u(x_s,t)}{\Gamma(1+\alpha)}.
$$

SOE technique

$$
\begin{aligned}
{}_a\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_s} &\approx \frac{1}{\Gamma(\alpha)}\left(\int_a^{x_{s-1}} \frac{\bar{u}(\xi,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi + \int_{x_{s-1}}^{x_s} \frac{\bar{u}(\xi,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi\right) \\
&\equiv \mathcal{I}_{\delta}^{a,s}(t) + \mathcal{I}_{\delta}^{s-1,s}(t),
\end{aligned}
$$

where

$$
\mathcal{I}_{\delta}^{s-1,s}(t) = \frac{1}{\Gamma(\alpha)}\int_{x_{s-1}}^{x_s} \frac{u(x_s,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi = \frac{({}_{s-1}\delta_s)^{\alpha} u(x_s,t)}{\Gamma(1+\alpha)}.
$$

SOE technique

$$
\begin{aligned}
{}_a\mathcal{I}_x^\alpha u(x,t)\Big|_{x_s} &\approx \frac{1}{\Gamma(\alpha)}\left(\int_a^{x_{s-1}} \frac{\bar{u}(\xi,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi + \int_{x_{s-1}}^{x_s} \frac{\bar{u}(\xi,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi\right)\\
&\equiv \mathcal{I}_\delta^{a,s}(t) + \mathcal{I}_\delta^{s-1,s}(t),
\end{aligned}
$$

where

$$
\mathcal{I}_\delta^{s-1,s}(t) = \frac{1}{\Gamma(\alpha)}\int_{x_{s-1}}^{x_s} \frac{u(x_s,t)}{(x_s-\xi)^{1-\alpha}}\mathrm{d}\xi = \frac{({}_{s-1}\delta_s)^\alpha u(x_s,t)}{\Gamma(1+\alpha)}.
$$

SOE technique

Lemma 1 (Jiang, S. et al., CiCP 2017)

For the given $\beta \in (0,1)$, tolerance error ϵ, $\Delta x > 0$, and a positive bound q, there are one positive integer N_{soe}, positive points λ_k and corresponding positive weights θ_k, $(k = 1, 2, \ldots, N_{soe})$ satisfying

$$\left| x^{-\beta} - \sum_{k=1}^{N_{soe}} \theta_k e^{-\lambda_k x} \right| \le \epsilon, \quad \forall x \in [\Delta x, q],$$

and the number of exponentials needed is of the order

$$N_{soe} = \mathcal{O}\left(\log\frac{1}{\epsilon}\left(\log\log\frac{1}{\epsilon} + \log\frac{q}{\Delta x}\right) + \log\frac{1}{\epsilon}\left(\log\log\frac{1}{\epsilon} + \log\frac{1}{\Delta x}\right)\right).$$

SOE technique

$$
\begin{aligned}
\mathcal{I}_\delta^{a,s}(t) &= \frac{1}{\Gamma(\alpha)} \int_a^{x_{s-1}} \bar{u}(\xi,t)\,(x_s-\xi)^{\alpha-1}\,\mathrm{d}\xi \\
&\approx \frac{1}{\Gamma(\alpha)} \int_a^{x_{s-1}} \bar{u}(\xi,t) \sum_{k=1}^{N_{soe}} \theta_k e^{-\lambda_k(x_s-\xi)}\mathrm{d}\xi \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{N_{soe}} \theta_k \int_a^{x_{s-1}} \bar{u}(\xi,t) e^{-\lambda_k(x_s-\xi)}\mathrm{d}\xi.
\end{aligned}
$$

$$
\begin{aligned}
v_{s,k}(t) &= e^{-\lambda_k({}_{s-1}\delta_s)} v_{s-1,k}(t) + \int_{x_{s-2}}^{x_{s-1}} \bar{u}(\xi,t) e^{-\lambda_k(x_s-\xi)}\mathrm{d}\xi \\
&= e^{-\lambda_k({}_{s-1}\delta_s)} v_{s-1,k}(t) + \frac{u(x_{s-1},t)}{\lambda_k}\left(e^{-\lambda_k({}_{s-1}\delta_s)} - e^{-\lambda_k({}_{s-2}\delta_s)}\right).
\end{aligned}
$$

SOE technique

$$
\begin{aligned}
\mathcal{I}_{\delta}^{a,s}(t) =& \frac{1}{\Gamma(\alpha)} \int_a^{x_{s-1}} \bar{u}(\xi, t)\, (x_s - \xi)^{\alpha-1}\, \mathrm{d}\xi \\
\approx& \frac{1}{\Gamma(\alpha)} \int_a^{x_{s-1}} \bar{u}(\xi, t) \sum_{k=1}^{N_{soe}} \theta_k e^{-\lambda_k (x_s - \xi)} \mathrm{d}\xi \\
=& \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{N_{soe}} \theta_k \int_a^{x_{s-1}} \bar{u}(\xi, t) e^{-\lambda_k (x_s - \xi)} \mathrm{d}\xi.
\end{aligned}
$$

$$
\begin{aligned}
v_{s,k}(t) =& e^{-\lambda_k ({}_{s-1}\delta_s)} v_{s-1,k}(t) + \int_{x_{s-2}}^{x_{s-1}} \bar{u}(\xi, t) e^{-\lambda_k (x_s - \xi)} \mathrm{d}\xi \\
=& e^{-\lambda_k ({}_{s-1}\delta_s)} v_{s-1,k}(t) + \frac{u(x_{s-1}, t)}{\lambda_k} \left(e^{-\lambda_k ({}_{s-1}\delta_s)} - e^{-\lambda_k ({}_{s-2}\delta_s)} \right).
\end{aligned}
$$

SOE technique

The fast approximation of the left Riemann-Liouville fractional integral:

$$ {}_a^{\delta}\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_s} = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{N_{soe}} \theta_k v_{s,k}(t) + \frac{({}_{s-1}\delta_s)^{\alpha} u(x_s,t)}{\Gamma(1+\alpha)}. $$

The fast approximation of the right Riemann-Liouville fractional integral:

$$ {}_x^{\delta}\mathcal{I}_b^{\alpha} u(x,t)\Big|_{x_s} = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{N_{soe}} \theta_k \tilde{v}_{m-s,k}(t) + \frac{({}_s\delta_{s+1})^{\alpha} u(x_s,t)}{\Gamma(1+\alpha)}, $$

with

$$ \tilde{v}_{s,k}(t) = e^{-\lambda_k({}_s\delta_{s+1})} \tilde{v}_{s-1,k}(t) + \frac{u(x_{m-s+1},t)}{\lambda_k} \left(e^{-\lambda_k({}_s\delta_{s+1})} - e^{-\lambda_k({}_s\delta_{s+2})} \right). $$

SOE technique

The fast approximation of the left Riemann-Liouville fractional integral:

$$ {}_a^{\delta}\mathcal{I}_x^{\alpha} u(x,t)\Big|_{x_s} = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{N_{soe}} \theta_k v_{s,k}(t) + \frac{({}_{s-1}\delta_s)^{\alpha} u(x_s,t)}{\Gamma(1+\alpha)}. $$

The fast approximation of the right Riemann-Liouville fractional integral:

$$ {}_x^{\delta}\mathcal{I}_b^{\alpha} u(x,t)\Big|_{x_s} = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{N_{soe}} \theta_k \tilde{v}_{m-s,k}(t) + \frac{({}_s\delta_{s+1})^{\alpha} u(x_s,t)}{\Gamma(1+\alpha)}, $$

with

$$ \tilde{v}_{s,k}(t) = e^{-\lambda_k({}_s\delta_{s+1})} \tilde{v}_{s-1,k}(t) + \frac{u(x_{m-s+1},t)}{\lambda_k} \left(e^{-\lambda_k({}_s\delta_{s+1})} - e^{-\lambda_k({}_s\delta_{s+2})} \right). $$

Fast implementation in matrix form

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_t(x,t)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)\mathrm{d}x \approx {}_{i-\frac{1}{2}}\delta_{i+\frac{1}{2}} u_t(x_i,t).$$

In matrix expression, we have

$$A\mathbf{u}^j = D\mathbf{u}^{j-1} + \tau\mathbf{f}^j, \quad j = 1, 2, \ldots, n,$$

where

$$A = D - \tau\gamma A^L - \tau(1-\gamma)A^R, \quad D = \operatorname{diag}\left(\left({}_{i-\frac{1}{2}}\delta_{i+\frac{1}{2}}\right)_{i=1}^{m-1}\right).$$

Fast implementation in matrix form

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_t(x,t)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)\mathrm{d}x \approx {}_{i-\frac{1}{2}}\delta_{i+\frac{1}{2}} u_t(x_i,t).$$

In matrix expression, we have

$$A\mathbf{u}^j = D\mathbf{u}^{j-1} + \tau\mathbf{f}^j, \quad j = 1, 2, \ldots, n,$$

where

$$A = D - \tau\gamma A^L - \tau(1-\gamma)A^R, \quad D = \mathrm{diag}\left(\left({}_{i-\frac{1}{2}}\delta_{i+\frac{1}{2}}\right)_{i=1}^{m-1}\right).$$

Fast implementation in matrix form

Algorithm 1 The fast matrix-vector multiplication $A^L\mathbf{z}$ for any vector $\mathbf{z}$

1: Compute the weights and nodes $\{\theta_k, \lambda_k\}$ of the SOE approximation
2: Set $v_{1,k} = 0$ and $length(\mathbf{z}) = m - 1$ denotes the size of the vector $\mathbf{z}$
3: **for** $s = 2 : m$ **do**
4: Compute $v_{s,k}$ by recursive formula $\mathcal{O}(m \log^2 m)$ complexity
5: **end for**
6: Compute ${}_a^{\delta}\mathcal{I}_x^{\alpha}\mathbf{z}\Big|_s$ $(s = 1, 2, \ldots, m)$ via SOE approxiamtion
7: $(A^L\mathbf{z})_i = \frac{\eta(x_{i+1/2})}{{}_i\delta_{i+1}}\left[{}_a^{\delta}\mathcal{I}_x^{\alpha}\mathbf{z}\Big|_{x_{i+1}} - {}_a^{\delta}\mathcal{I}_x^{\alpha}\mathbf{z}\Big|_{x_i}\right] - \frac{\eta(i-1/2)}{{}_{i-1}\delta_i}\left[{}_a^{\delta}\mathcal{I}_x^{\alpha}\mathbf{z}\Big|_{x_i} - {}_a^{\delta}\mathcal{I}_x^{\alpha}\mathbf{z}\Big|_{x_{i-1}}\right]$

Error analysis

Lemma 2

Suppose $u(x,t) \in C^1[a,b]$ with respect to x. The difference between the left Riemann-Liouville fractional integral at point x_s and its fast approximation formula can be expressed as

$$\left| {}^{\delta}_{a}\mathcal{I}^{\alpha}_{x} u(x,t)\Big|_{x_s} - {}_{a}\mathcal{I}^{\alpha}_{x} u(x,t)\Big|_{x_s} \right| \leq \mathcal{O}(\delta_{\max} + \epsilon), \quad s = 1, 2, \ldots, m,$$

where ϵ denotes the accuracy of the SOE approximation. Analogously, for the right Riemann-Liouville fractional integral, it holds that

$$\left| {}^{\delta}_{x}\mathcal{I}^{\alpha}_{b} u(x,t)\Big|_{x_s} - {}_{x}\mathcal{I}^{\alpha}_{b} u(x,t)\Big|_{x_s} \right| \leq \mathcal{O}(\delta_{\max} + \epsilon), \quad s = 0, 1, \ldots, m-1.$$

Stability analysis

$$A\mathbf{u}^j = D\mathbf{u}^{j-1} + \tau\mathbf{f}^j, \quad A = D - \tau\gamma A^L - \tau(1-\gamma)A^R.$$

Lemma 3

Assume that $\eta(x)$ satisfies the Lipschitz condition. The matrices A^L and A^R are strictly row-wise diagonally dominant with negative diagonals and non-negative sub-/super-diagonals for sufficiently accurate SOE approximation.

Theorem 4

The fast scheme is unconditionally stable with respect to the initial condition.

Stability analysis

$$A\mathbf{u}^j = D\mathbf{u}^{j-1} + \tau\mathbf{f}^j, \quad A = D - \tau\gamma A^L - \tau(1-\gamma)A^R.$$

Lemma 3

Assume that $\eta(x)$ satisfies the Lipschitz condition. The matrices A^L and A^R are strictly row-wise diagonally dominant with negative diagonals and non-negative sub-/super-diagonals for sufficiently accurate SOE approximation.

Theorem 4

The fast scheme is unconditionally stable with respect to the initial condition.

Outline

Banded preconditioner

$$\tilde{A} = D - \tau\bar{\eta}\gamma H^L - \tau\bar{\eta}(1-\gamma)H^R,$$

where

$$\left\{h^L_{i,j}\right\} = \frac{1}{\Gamma(1+\alpha)} \begin{cases} 0, & j > i+1, \\ ({}_i\delta_{i+1})^{\alpha-1}, & j = i+1, \\ -({}_i\delta_{i+1})^{-1}\Big(({}_i\delta_{i+1})^{\alpha} + ({}_{i-1}\delta_i)^{\alpha} - ({}_{i-1}\delta_{i+1})^{\alpha}\Big) + ({}_{i-1}\delta_i)^{\alpha-1}, & j = i, \\ -({}_i\delta_{i+1})^{-1}\Big(({}_j\delta_{i+1})^{\alpha} + ({}_{j-1}\delta_i)^{\alpha} - ({}_{j-1}\delta_{i+1})^{\alpha} + ({}_j\delta_i)^{\alpha}\Big) & \\ +({}_{i-1}\delta_i)^{-1}\Big(({}_j\delta_i)^{\alpha} + ({}_{j-1}\delta_{i-1})^{\alpha} - ({}_{j-1}\delta_i)^{\alpha} + ({}_j\delta_{i-1})^{\alpha}\Big), & j \le i-1, \end{cases}$$

$$\left\{h^R_{i,j}\right\} = \frac{1}{\Gamma(1+\alpha)} \begin{cases} ({}_i\delta_{i+1})^{-1}\Big(({}_{i+1}\delta_{j+1})^{\alpha} + ({}_i\delta_j)^{\alpha} - ({}_{i+1}\delta_j)^{\alpha} + ({}_i\delta_{j+1})^{\alpha}\Big) & \\ -({}_{i-1}\delta_i)^{-1}\Big(({}_i\delta_{j+1})^{\alpha} + ({}_{i-1}\delta_j)^{\alpha} - ({}_i\delta_j)^{\alpha} + ({}_{i-1}\delta_{j+1})^{\alpha}\Big), & j \ge i+1, \\ -({}_i\delta_{i+1})^{-1}\Big(({}_i\delta_{i+1})^{\alpha} + ({}_{i-1}\delta_i)^{\alpha} - ({}_{i-1}\delta_{i+1})^{\alpha}\Big) + ({}_{i-1}\delta_i)^{\alpha-1}, & j = i, \\ ({}_{i-1}\delta_i)^{\alpha-1}, & j = i-1, \\ 0, & j < i-1. \end{cases}$$

Banded preconditioner

$$P = D - \tau\bar{\eta}\gamma B^L - \tau\bar{\eta}(1-\gamma)B^R,$$

where B^L and B^R are the main $\ell + 1$ bands of the Heisenberg matrices H^L and H^R, respectively.

Outline

PGMRES method
stopping criterion: $\frac{\|r_k\|_2}{\|b\|_2} < 10^{-6}$

Example 1

True solution is given by $u(x,t) = 4e^{-t}x(2-x)$, for $0 \le x \le 2$ and $0 \le T \le 1$. The diffusivity coefficient $\eta = 1$ and the source function is given by

$$f(x,t) = -4e^{-t}x(2-x) - 8e^{-t}\Big[\gamma\Big(\frac{x^{\alpha-1}}{\Gamma(\alpha)} - \frac{x^{\alpha}}{\Gamma(1+\alpha)}\Big) + (1-\gamma)\Big(\frac{(2-x)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(2-x)^{\alpha}}{\Gamma(1+\alpha)}\Big)\Big].$$

PGMRES method
stopping criterion: $\frac{\|r_k\|_2}{\|b\|_2} < 10^{-6}$

Example 1

True solution is given by $u(x,t) = 4e^{-t}x(2-x)$, for $0 \le x \le 2$ and $0 \le T \le 1$. The diffusivity coefficient $\eta = 1$ and the source function is given by

$$f(x,t) = -4e^{-t}x(2-x) - 8e^{-t}\left[\gamma\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)} - \frac{x^{\alpha}}{\Gamma(1+\alpha)}\right) + (1-\gamma)\left(\frac{(2-x)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(2-x)^{\alpha}}{\Gamma(1+\alpha)}\right)\right].$$

$$x_j = \begin{cases} (b-a)\left(\frac{j}{s}\right)^r, & j = 0, 1, \ldots, s, \text{ and } s = \frac{m}{2}, \\ b - a - x_{m-j}, & j = s+1, s+2, \ldots, m, \end{cases}$$

where m is set as an even number and $r = \frac{1}{1-\alpha} > 1$.

Table: Numerical results for Example 1 with fixed $n = 2^{11}$ and $\alpha = 0.5$.

m	Gauss Elimination			Fast GMRES				Fast PGMRES			
	CPU	Error	Order	Iter	CPU	Error	Order	Iter	CPU	Error	Order
2^7	3.91	2.8648e-2	–	17.00	1.31	2.9338e-2	–	3.31	0.69	2.9232e-2	–
2^8	20.50	1.4385e-2	0.99	25.00	4.98	1.4734e-2	0.99	4.01	1.69	1.4731e-2	0.99
2^9	218.73	7.0891e-3	1.02	42.00	17.35	7.2598e-3	1.02	6.00	3.94	7.2348e-3	1.03
2^{10}	2079.16	3.3950e-3	1.06	74.00	67.84	3.4775e-3	1.06	8.00	8.87	3.4884e-3	1.05
2^{11}	–	–	–	128.00	448.56	1.5741e-3	1.14	13.00	45.18	1.5904e-3	1.13

Example 2

We consider Example 1 with the diffusivity coefficient function $\eta(x) = 10 + x(2-x)$. The relevant source function is derived as

$$\begin{aligned} f(x,t) = & -4e^{-t}x(2-x) \\ & -\frac{8e^{-t}}{\Gamma(\alpha+2)}\Big(\gamma x^{\alpha-1}W(x) + (1-\gamma)(2-x)^{\alpha-1}W(2-x)\Big), \end{aligned}$$

where
$W(z) = 10\alpha(\alpha+1) - (\alpha+1)(2\alpha-8)z - (\alpha+2)(\alpha+3)z^2 + (\alpha+3)z^3.$

Table: Numerical results for Example 2 with fixed $n = 4m$ and $\alpha = 0.5$.

m	Fast GMRES				Fast PGMRES			
	Iter	CPU	Error	Order	Iter	CPU	Error	Order
2^6	26.00	0.38	5.6992e-2	–	7.89	0.14	5.6757e-2	–
2^7	40.00	1.13	2.9562e-2	0.95	7.41	0.45	2.9456e-2	0.95
2^8	58.00	8.15	1.5071e-2	0.97	8.03	1.69	1.5060e-2	0.97
2^9	84.00	78.74	7.6122e-3	0.99	9.01	6.48	7.4403e-3	1.02
2^{10}	119.00	302.95	3.8260e-3	0.99	12.00	31.99	3.9200e-3	0.92

Example 3

$$\begin{cases} u_t - \left({}_0\mathcal{D}_x^{2-\alpha}\right) u = f(x,t), \ (x,t) \in [0,1] \times (0,1], \\ u(0,t) = 0, u(1,t) = e^t, t \in [0,1], \\ u(x,0) = x^\beta. \end{cases}$$

The exact solution is set by $u(x,t) = e^t x^\beta$. The source function is correspondingly derived as

$$f(x,t) = e^t x^\beta - \frac{e^t \Gamma(1+\beta)}{\Gamma(\alpha+\beta-1)} x^{\alpha+\beta-2}.$$

$$x_j = (b-a)\left(\frac{j}{m}\right)^r, \; j = 0, 1, \ldots, m, \; \text{with } r = \frac{2}{1-\alpha} > 1.$$

Table: Numerical results for Example 2 with fixed $n = m$ and $\alpha = 0.8, \beta = 0.5$.

m	Gauss Elimination			Fast GMRES				Fast PGMRES			
	CPU	Error	Order	Iter	CPU	Error	Order	Iter	CPU	Error	Order
2^7	0.24	1.2710e-2	–	90.98	1.41	1.2710e-2	–	6.00	0.17	1.2711e-2	–
2^8	2.53	6.3773e-3	0.99	156.37	9.58	6.3793e-3	0.99	9.00	0.71	6.3765e-3	1.00
2^9	49.03	3.1965e-3	1.00	286.32	98.79	3.1976e-3	1.00	12.00	3.67	3.2039e-3	0.99
2^{10}	1035.49	1.6009e-3	1.00	532.63	1390.77	1.6048e-3	0.99	19.00	39.94	1.5988e-3	1.00
2^{11}	–	–	–	–	–	–	–	29.00	272.76	8.0865e-4	0.98

Remarks

- Fast implementation for high dimensional problems?
- Convergence analysis (first-order accuracy)?

Thanks for your attention!