Fractional collocation method for third-kind Volterra integral equations with nonsmooth solutions

Chengming Huang
(Joint work with: Zheng Ma)

School of Mathematics and Statistics
Huazhong University of Science and Technology

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Overview

Overview

Third-kind VIEs

- Third-kind Volterra integral equations (VIEs) of the form

$$a(t)u(t) = g(t) + \int_0^t (t-s)^{-\mu} K(t,s)u(s)ds, \quad t \in I := [0,T], \quad (1)$$

with $a(t) = t^\gamma$ ($\gamma > 0$), $0 \le \mu < 1$, $g(t) \in C(I)$, $K(t,s) \in C(D)$, $D := \{(t,s) : 0 \le s \le t \le T\}$.

- $a(t) \equiv 0$: first-kind VIE; $a(t) \ne 0$ for all $t \in I$: second-kind VIE.

- The equation (1) can be written equivalently as

$$u(t) = g_1(t) + (K_{\mu,\gamma}u)(t), \quad t \in I := [0,T],$$

where $g_1(t) = t^{-\gamma}g(t)$ and

$$(K_{\mu,\gamma}u)(t) := t^{-\gamma}\int_0^t (t-s)^{-\mu} K(t,s)u(s)\,ds.$$

Operator $K_{\mu,\gamma}$

$$(K_{\mu,\gamma}u)(t) = t^{-\gamma}\int_0^t (t-s)^{-\mu} K(t,s)u(s)\,ds$$

- When $\gamma < 1-\mu$, $K_{\mu,\gamma}$ is bounded and compact.
- When $\gamma = 1-\mu$,

$$\begin{aligned} K_{\mu,\gamma}u(t) &= t^{\mu-1}\int_0^t (t-s)^{-\mu} K(t,s)u(s)\,ds \\ &= \int_0^t t^{-1}\phi(s/t)K(t,s)u(s)\,ds, \end{aligned}$$

where $\phi(x) = (1-x)^{-\mu} \in L^1(0,1)$. In this case, $K_{\mu,\gamma}$ is a cordial Volterra integral operator (Vainikko, 2009).

Operator $K_{\mu,\gamma}$

$$(K_{\mu,\gamma}u)(t) = t^{-\gamma}\int_0^t (t-s)^{-\mu}K(t,s)u(s)\,ds$$

- When $\gamma > 1-\mu$, if $K(t,s) = s^{\gamma+\mu-1}K_1(t,s)$ with $K_1(t,s) \in C(D)$,

$$\begin{aligned} K_{\mu,\gamma}u(t) &= t^{-\gamma}\int_0^t (t-s)^{-\mu}K(t,s)u(s)\,ds \\ &= t^{-\gamma}\int_0^t (t-s)^{-\mu}s^{\gamma+\mu-1}K_1(t,s)u(s)\,ds \\ &= \int_0^t t^{-1}\psi(s/t)K_1(t,s)u(s)\,ds, \end{aligned}$$

where $\psi(x) = (1-x)^{-\mu}x^{\gamma+\mu-1} \in L^1(0,1)$. In this case, $K_{\mu,\gamma}$ is a cordial Volterra integral operator.

Theoretical properties

Theoretical analysis of third-kind VIEs:

- G. C. Evans., Trans. Amer. Math. Soc., 1910;
- T. Sato, J. Math. Soc. Japan, 1953;
- W. Han, J. Integral Equations Appl., 1994;
- P. Grandits, J. Integral Equations Appl., 2008;
- G. Vainikko. Numer. Funct. Anal. Optim., 2009/2010/2011;
- S. S. Allaei, Z. Yang, H. Brunner, J. Integral Equations Appl., 2015;
- H. Brunner, 2017;
- ......

Existence and uniqueness of the exact solution

Consider

$$u(t) = g_1(t) + (K_{\mu,\gamma}u)(t), \quad t \in I := [0, T], \tag{2}$$

with

$$(K_{\mu,\gamma}u)(t) := t^{-\gamma}\int_0^t (t-s)^{-\mu} s^{\mu+\gamma-1} K_1(t,s)u(s)\,ds.$$

Lemma 1.1 (Allaei, Yang, Brunner, JIEA, 2015)

If $g_1(t) \in C(I)$, $K_1(t,s) \in C(D)$ with $D = \{(t,s) : 0 \le s \le t \le T\}$, $K_1(0,0) \neq 0$, then $K_{\mu,\gamma}$ is a non-compact operator with the spectrum

$$\sigma_{C(I)}(K_{\mu,\gamma}) = \{0\} \cup \{K_1(0,0)B(\Lambda+\mu+\gamma, 1-\mu) : \Lambda \in \mathbb{C}, Re\,\Lambda \ge 0\}. \tag{3}$$

If $1 \notin \sigma_{C(I)}(K_{\mu,\gamma})$, then equation (2) has a unique solution $u \in C(I)$.

Numerical methods

$$u(t) = g_1(t) + t^{-\gamma} \int_0^t (t-s)^{-\mu} s^{\mu+\gamma-1} K_1(t,s) u(s)\, ds, \tag{2'}$$

- for $\mu = 0$
    - T. Diogo, S. McKee, T. Tang., Hermite-type collocation method, IMA J. Numer. Anal., 1991;
    - T. Diogo, N. B. Franco, P. Lima., High order product integration methods, Pure Appl. Anal., 2004;
    - T. Diogo, P. Lima., Superconvergence of collocation methods, J. Comput. Appl. Math., 2008;
    - J. Ma, Y. Jiang. On a graded mesh method, J. Comput. Appl. Math., 2009;
    - Y. Yang, Z. Tang., Mapped spectral collocation methods, Appl. Numer. Math., 2021;
    - H. Song, Z. Yang, Y. Xiao, Superconvergence analysis of collocation methods, Appl. Math. Comput., 2022;
    - ......

Numerical methods

$$u(t) = g_1(t) + t^{-\gamma} \int_0^t (t-s)^{-\mu} s^{\mu+\gamma-1} K_1(t,s) u(s)\, ds, \qquad (2')$$

- for $\mu > 0$
  - Pereverzev, Prossdorf, Piecewise constant approximate, J. Inverse Ill-Posed Probl., 1997;
  - Vainikko, Spline collocation-interpolation method, Numer. Funct. Anal. Optim., 2011;
  - Allaei, Yang, Brunner, Collocation methods, IMA JNA 2017;
  - Song, Yang, Brunner, Collocation methods for nonlinear equations, Calcolo, 2019;
  - Cai, Legendre-Galerkin methods, J. Sci. Comput., 2020;
  - Ma, Huang, Spectral collocation method, JCAM, 2021;
  - Wang, Zhou, Guo, hp collocation method, J. Sci. Comput., 2021;
- All the above methods are based on polynomial approximation, and in most theoretical analysis, the exact solution is assumed to be sufficiently smooth.

Aim of our work

- In this work, we consider

$$u(t) = g_1(t) + (K_{\mu,\gamma}u)(t), \quad t \in I := [0, T], \tag{2}$$

with $\gamma \geq 1-\mu$, where $g_1(t) = t^{-\gamma}g(t)$ and

$$(K_{\mu,\gamma}u)(t) := t^{-\gamma}\int_0^t (t-s)^{-\mu} s^{\mu+\gamma-1} K_1(t,s)u(s)\,ds.$$

- The operator $K_{\mu,\gamma}$ is bounded but non-compact if $K_1(0,0) \neq 0$.
- The solution is admitted to be weakly singular at $t = 0$.
- Aim of our work
  - Fractional collocation method for (2)
  - Solvability and error analysis of the proposed method

Overview

Overview

Preliminaries

- The partition $\mathscr{T}_M$ of the interval I:

$$\mathscr{T}_M := \{t_i : 0 = t_0 < t_1 < \cdots < t_M = T\}$$

  and set $\sigma_i = (t_{i-1}, t_i], h_i = t_i - t_{i-1}$.
- The approximate space:

$$S_\lambda(\mathscr{T}_M) := \{v(t) : v(t)|_{t\in\sigma_i} \in P_N^\lambda,\ i = 1, \cdots, M\},$$

  where $0 < \lambda \leq 1$ and

$$P_N^\lambda := span\{1, t^\lambda, \ldots, t^{N\lambda}\}.$$

Preliminaries

- Let $\sigma_{i,\lambda} := (t_{i-1}^{\lambda}, t_i^{\lambda}]$, $h_{i,\lambda} = t_i^{\lambda} - t_{i-1}^{\lambda}$, $0 \le c_0 < \cdots < c_N \le 1$ and

$$\xi_{i,k} = t_{i-1}^{\lambda} + c_k h_{i,\lambda} \quad \text{for } i = 1, \ldots, M, \; k = 0, \ldots, N.$$

- The set of collocation points $X_{\mathscr{T}}$

$$X_{\mathscr{T}} := \{t_{i,k} : t_{i,k} = \rho(\xi_{i,k}), \; k = 0, \ldots, N, \; i = 1, \ldots, M\},$$

where $\rho(s) = s^{1/\lambda}$.

- The fractional interpolation basis functions $L^{\lambda}_{i,k}(t)$

$$L^{\lambda}_{i,k}(t) = \prod_{j=0, j\neq k}^{N} \frac{t^{\lambda} - t^{\lambda}_{i,j}}{t^{\lambda}_{i,k} - t^{\lambda}_{i,j}} \quad \text{for} \quad t \in \sigma_i, \; i = 1, \cdots, M.$$

- The fractional interpolation operator $I^{\lambda}_{N,i} : C(\sigma_i) \to P^{\lambda}_N(\sigma_i)$

$$(I^{\lambda}_{N,i} v)(t) := \sum_{k=0}^{N} L^{\lambda}_{i,k}(t) v(t_{i,k}) \quad \text{for} \quad t \in \sigma_i, \; i = 1, \cdots, M.$$

- For any $v \in P^{\lambda}_N(\sigma_i)$,

$$v(t) = \sum_{k=0}^{N} L^{\lambda}_{i,k}(t) v(t_{i,k}) = (I^{\lambda}_{N,i} v)(t) \quad \text{for} \quad t \in \sigma_i, \; i = 1, \cdots, M.$$

Preliminaries

- The polynomial interpolation basis functions $L_{i,k}(s)$

$$L_{i,k}(s) = \prod_{j=0, j\neq k}^{N} \frac{s-\xi_{i,j}}{\xi_{i,k}-\xi_{i,j}} \quad \text{for} \quad s \in \sigma_{i,\lambda}, \; i = 1, \cdots, M.$$

- The polynomial interpolation operator $I_{N,i} : C(\sigma_{i,\lambda}) \to P_N(\sigma_{i,\lambda})$

$$(I_{N,i}w)(s) := \sum_{k=0}^{N} L_{i,k}(s) w(\xi_{i,k}) \quad \text{for} \quad s \in \sigma_{i,\lambda}, \; i = 1, \cdots, M.$$

By $t = \rho(s) = s^{1/\lambda}$, for $t \in \sigma_i$ (which implies $s \in \sigma_{i,\lambda}$), one has

$$L_{i,k}^{\lambda}(t) = \prod_{j=0, j\neq k}^{N} \frac{t^{\lambda} - t_{i,j}^{\lambda}}{t_{i,k}^{\lambda} - t_{i,j}^{\lambda}} = \prod_{j=0, j\neq k}^{N} \frac{s - \xi_{i,j}}{\xi_{i,k} - \xi_{i,j}} = L_{i,k}(s)$$

and

$$(I_{N,i}^{\lambda} v)(t) = \sum_{k=0}^{N} L_{i,k}^{\lambda}(t) v(t_{i,k}) = \sum_{k=0}^{N} L_{i,k}(s) v(\rho(\xi_{i,k})) = (I_{N,i} \hat{v})(s),$$

where $\hat{v}(s) = v(\rho(s))$.

Fractional collocation method

The fractional collocation method for (2)

Find a function $U \in S_\lambda(\mathscr{T}_M)$ such that for $i = 1, \ldots, M$,

$$U(t_{i,k}) = g_1(t_{i,k}) + t_{i,k}^{-\gamma} \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} (t_{i,k} - s)^{-\mu} s^{\gamma+\mu-1} I_{N,j}^{\lambda}(K_1(t_{i,k}, s) U_j(s)) ds$$
$$+ t_{i,k}^{-\gamma} \int_{t_{i-1}}^{t_{i,k}} (t_{i,k} - s)^{-\mu} s^{\gamma+\mu-1} I_{N,i}^{\lambda}(K_1(t_{i,k}, s) U_i(s)) ds, \quad k = 0, \cdots, N, \tag{4}$$

The numerical solution has the local representation

$$U_i(t) = \sum_{k=0}^{N} L_{i,k}^{\lambda}(t) U(t_{i,k}) \ \text{ for } \ t \in \sigma_i, \ i = 1, \ldots, M.$$

$$u(t) = g_1(t) + t^{-\gamma} \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} (t - s)^{-\mu} s^{\gamma+\mu-1} (K_1(t, s) u(s)) ds$$
$$+ t^{-\gamma} \int_{t_{i-1}}^{t} (t - s)^{-\mu} s^{\gamma+\mu-1} (K_1(t, s) u(s)) ds, \quad t \in \sigma_i.$$

Rewrite the scheme (4) as follow

$$U_{i,k} = g_{1,i,k} + \sum_{j=1}^{i} \sum_{l=0}^{N} \phi_{k,l}^{i,j} U_{j,l}, \;\; k = 0, \cdots, N,$$

where $U_{i,k} = U(t_{i,k})$, $g_{1,i,k} = g_1(t_{i,k})$ and

$$\phi_{k,l}^{i,j} = \begin{cases} t_{i,k}^{-\gamma} \int_{t_{j-1}}^{t_j} (t_{i,k} - s)^{-\mu} s^{\gamma+\mu-1} L_{j,l}^{\lambda}(s) ds K_1(t_{i,k}, t_{j,l}), & 1 \le j \le i-1, \\ t_{i,k}^{-\gamma} \int_{t_{i-1}}^{t_{i,k}} (t_{i,k} - s)^{-\mu} s^{\gamma+\mu-1} L_{i,l}^{\lambda}(s) ds K_1(t_{i,k}, t_{i,l}), & j = i. \end{cases}$$

Let $U_i := (U_{i,0}, \cdots, U_{i,N})^T$, $G_i := (g_{1,i,0}, \cdots, g_{1,i,N})^T$ and

$$\Phi^{i,j} := \begin{pmatrix} \phi_{0,0}^{i,j} & \cdots & \phi_{0,N}^{i,j} \\ \vdots & \ddots & \vdots \\ \phi_{N,0}^{i,j} & \cdots & \phi_{N,N}^{i,j} \end{pmatrix}, \quad 1 \le j \le i.$$

The matrix form of the scheme

$$(\mathbb{I}_{N+1} - \Phi^{i,i})U_i = G_i + \sum_{j=1}^{i-1} \Phi^{i,j} U_j, \tag{5}$$

where $\mathbb{I}_{N+1}$ is the identity matrix of order $N+1$.

Overview

On uniform mesh and graded mesh

For example, let $\mu = 0$, $\gamma = 1$, $K_1(t, s) = K_0$.

- When $t_i = T(i/M)$ for $i = 1, \ldots, M$. For $K_0 = 5, N = 0, c_0 = 1/2$,

$$\mathbb{I}_{N+1} - \Phi^{i,i} = 1 - \frac{5}{2i+1}.$$

- When $t_i = T(i/M)^q$ $(q > 1)$ for $i = 1, \ldots, M$. For $K_0 = 9/5$, $N = 0$, $c_0 = 1$, $q = 2$,

$$\mathbb{I}_{N+1} - \Phi^{i,i} = 1 - K_0 \frac{(i+1)^2 - i^2}{(i+1)^2}.$$

In these two cases, when $i = 2$, $\mathbb{I}_{N+1} - \Phi^{i,i}$ is singular for any M. So the system (5) is not well-posed.

The design of mesh

Set

$$t_0 = 0, \quad t_1 = T\left(\frac{1}{M}\right)^p,$$
$$t_i = \left(t_1^{\frac{1}{q}} + (T^{\frac{1}{q}} - t_1^{\frac{1}{q}})\frac{i-1}{M-1}\right)^q, \ 2 \le i \le M, \tag{6}$$

- When $p = q$, this is the graded mesh with grading exponent p, namely $t_i = T(i/M)^p$.
- When $p = q = 1$, this is the uniform mesh, namely $t_i = T(i/M)$.
- Let $1 \le p < q$, the solvability of the scheme can be proved, by using some similar techniques in [Allaei, Yang, Brunner, IMA J.Numer. Anal, 2017].

Solvability in the first subinterval

$$\begin{aligned}\Phi^{i,i}_{k,l} &= t_{i,k}^{-\gamma}\int_{t_{i-1}}^{t_{i,k}}(t_{i,k}-s)^{-\mu}s^{\gamma+\mu-1}L^{\lambda}_{i,l}(s)dsK_1(t_{i,k},t_{i,l})\\ &= \int_0^1(1-\theta)^{-\mu}\theta^{\gamma+\mu-1}L^{\lambda}_{i,l}(t_{i,k}\theta)d\theta K_1(t_{i,k},t_{i,l})\ \text{(for i=1)}\end{aligned}$$

For $i=1$, define

$$\psi_{k,l}=\int_0^1(1-\theta)^{-\mu}\theta^{\gamma+\mu-1}L_l(c_k\theta^{\lambda})d\theta,\quad k,l=0,\cdots,N.$$

Then

$$\Phi^{1,1}=K_1(0,0)\Psi+\widetilde{\Psi},$$

where

$$\begin{aligned}\Psi &:= \left(\psi_{k,l}\right)_{k,l=0,\cdots,N}\\ \widetilde{\Psi} &:= \left(\left(K_1(t_{1,k},t_{1,l})-K_1(0,0)\right)\psi_{k,l}\right)_{k,l=0,\cdots,N}.\end{aligned}$$

Solvability in the first subinterval

For matrix $\Psi := \left(\psi_{k,l}\right)_{k,l=0,\cdots,N}$,

$$\psi_{k,l} = \int_0^1 (1-\theta)^{-\mu} \theta^{\gamma+\mu-1} L_l(c_k \theta^{\lambda}) d\theta, \quad k,l = 0,\cdots,N,$$
$$\mathbb{I}_{N+1} - \Phi^{1,1} = \mathbb{I}_{N+1} - K_1(0,0)\Psi - \widetilde{\Psi},$$

the following result holds.

Lemma 2.1

For any given distinct collocation parameters $0 < c_0 < \cdots < c_N \leq 1$, one has

$$\Psi = VSV^{-1},$$

where

$$V = \left(c_k^n\right)_{k,n=0,\cdots,N}, \qquad S = \mathrm{diag}\big(B(1-\mu, \gamma+\mu+\lambda n)\big)_{n=0,\cdots,N}. \qquad (7)$$

Solvability in the first subinterval

For matrix

$$\widetilde{\Psi} := \left(\left(K_1(t_{1,k}, s_{1,l}) - K_1(0,0) \right) \psi_{k,l} \right)_{k,l=0,\cdots,N} = \Phi^{1,1} - K_1(0,0)\Psi,$$

the following result holds.

Lemma 2.2

Assume that the function K_1 is continuous and satisfies[a]

$$K_1(0,0) \neq \frac{1}{B(1-\mu, \gamma+\mu+\lambda n)}, \quad n = 0, \cdots, N.$$

Then there exists $\tilde{h} > 0$ such that for $h_1 < \tilde{h}$,

$$\|\widetilde{\Psi}\|_\infty \leq \frac{1}{2\|(\mathbb{I}_{N+1} - K_1(0,0) VSV^{-1})^{-1}\|_\infty}.$$

[a] This condition can be guaranteed by the condition of existence and uniqueness of the exact solution. This is not an "extra" condition.

Solvability in the first subinterval

Theorem 2.1

Assume that the function K_1 *is continuous and satisfies*

$$K_1(0,0) \neq \frac{1}{B(1-\mu, \gamma+\mu+\lambda n)}, \quad n = 0, \cdots, N.$$

Then the matrix $\mathbb{I}_{N+1} - \Phi^{1,1}$ *is invertible and*

$$\|(\mathbb{I}_{N+1} - \Phi^{1,1})^{-1}\|_\infty \leq 2\|(\mathbb{I}_{N+1} - K_1(0,0)VSV^{-1})^{-1}\|_\infty$$

whenever $h_1 > 0$ *is sufficiently small.*

Solvability in the other subintervals

Lemma 2.3

The modified graded mesh (6) has the property that for any given $\varepsilon > 0$*, there exists an* $M_0 > 0$ *such that for all* $M > M_0$ *there holds*

$$\max_{2 \le i \le M} \frac{h_i}{t_{i-1}} \le \varepsilon.$$

Theorem 2.2

There exists M_0 *such that for modified mesh (6), the matrix* $\mathbb{I}_{N+1} - \Phi^{i,i}$ *are invertible and*

$$\max_{2 \le i \le M} \left\| (\mathbb{I}_{N+1} - \Phi^{i,i})^{-1} \right\|_\infty \le 2$$

whenever $M > M_0$*.*

Overview

Lemma 2.4

For $f \in C^m(\sigma_{i,\lambda})$ *with* $1 \le m \le N+1$,

$$\|f(t) - I_{N,i}f(t)\|_{L^\infty(\sigma_{i,\lambda})} \le c h_{i,\lambda}^m \|f^{(m)}(t)\|_{L^\infty(\sigma_{i,\lambda})}.$$

Theorem 2.3

Let u *be the exact solution of equation* (1) *and* U *be the solution of scheme* (4) *with mesh* (6). *Assume that* $K_1(t, s^{1/\lambda}) \in C^m(I \times I_\lambda)$, $u(t^{1/\lambda}) \in C^m(I_\lambda)$ *with* $1 \le m \le N+1$, *where* $I_\lambda = [0, T^\lambda]$. *Then we have*

$$\|u - U\|_{L^\infty(I)} \le cM^{-\min\{p\lambda,1\}m} \left\| \partial_s^m (K_1(t, s^{1/\lambda}) u(s^{1/\lambda})) \right\|_{L^\infty(I, L^\infty(I_\lambda))}.$$

Overview

- Mesh

$$t_0 = 0, \quad t_1 = T\left(\frac{1}{M}\right)^p,$$
$$t_i = \left(t_1^{\frac{1}{q}} + (T^{\frac{1}{q}} - t_1^{\frac{1}{q}})\frac{i-1}{M-1}\right)^q, \ 2 \le i \le M,$$

- Collocation points

$$t_{i,k} = \left(t_{i-1}^{\lambda} + c_k(t_i^{\lambda} - t_{i-1}^{\lambda})\right)^{1/\lambda}, \ k = 0, 1, \ldots, N,$$

with $c_k = (k+1)/(N+2)$.

- Let $e_M = \|u - U\|_{L^\infty(I)}$, $r = \log_2(e_M/e_{2M})$.
- Let "EOC" be the expected order of convergence predicted by theoretical analysis.

Example 1

Example 1

Consider the linear VIE

$$t^{\gamma}u(t) = g(t) + \int_0^t \frac{\sqrt{3}}{3\pi}(t-s)^{-\mu}s^{\gamma+\mu-1}u(s)ds, \quad t \in [0,1],$$

where $g(t)$ is given function such that the exact solution of this problem is $u(t) = \sin(t^a) + \cos(t^a)$.

- The exact solution u has a weak singularity at $t = 0$ for $a \in (0,1)$. In this example, we set $a = 1/2$, $\gamma = 2/3$ and $\mu = 2/3$.
- Take $\lambda = 1/2$. In this case, $u(t^{1/\lambda})$ is analytic.

Table 1: The convergence of polynomial collocation for Example 1.

M	$\lambda = 1$ $(p = 1, q = 2)$					
	$N = 1$		$N = 2$		$N = 3$	
	e_N	r	e_N	r	e_N	r
64	2.34E-02	0.50	1.26E-02	0.50	7.82E-03	0.50
128	1.66E-02	0.50	8.92E-03	0.50	5.53E-03	0.50
256	1.17E-02	0.50	6.30E-03	0.50	3.91E-03	0.50
EOC		0.5		0.5		0.5

- Since the solution exhibits weak singularity of t^a $(a = 1/2)$, the order of convergence of polynomial collocation is no more than $1/2$.

Table 2: The convergence of fractional collocation for Example 1.

M	$\lambda = 1/2$ $(p=1, q=2)$					
	$N=1$		$N=2$		$N=3$	
	e_N	r	e_N	r	e_N	r
64	1.67E-03	1.04	2.46E-05	1.44	3.70E-07	2.03
128	8.17E-04	1.03	8.92E-06	1.46	9.08E-08	2.02
256	4.02E-04	1.02	3.21E-06	1.47	2.24E-08	2.02
EOC		1		1.5		2

- Fractional collocation with $\lambda = 1/2$ shows higher order than polynomial collocation and has no order barrier under same mesh parameters, since $u(t^{1/\lambda})$ has better regularity.

Table 3: The convergence of polynomial collocation for Example 1.

M	$\lambda = 1\ (p = 2, q = 3)$					
	$N = 1$		$N = 2$		$N = 3$	
	e_N	r	e_N	r	e_N	r
64	2.93E-03	1.00	1.58E-03	1.00	9.77E-04	1.00
128	1.46E-03	1.00	7.88E-04	1.00	4.89E-04	1.00
256	7.31E-04	1.00	3.94E-04	1.00	2.44E-04	1.00
EOC		1		1		1

- The order of convergence of polynomial collocation can be improved by using an appropriate mesh.

Table 4: The convergence of fractional collocation for Example 1.

M	$\lambda = 1/2$ $(p = 2, q = 3)$					
	$N = 1$		$N = 2$		$N = 3$	
	e_N	r	e_N	r	e_N	r
64	4.39E-05	1.97	5.28E-08	2.96	5.65E-10	3.83
128	1.16E-05	1.98	6.73E-09	2.97	3.84E-11	3.88
256	2.82E-06	1.98	8.56E-10	2.98	3.57E-12	3.43
EOC		2		3		4

- Fractional collocation with $\lambda = 1/2$ shows optimal convergence order under appropriate mesh parameters.

Example 2

Example 2

Consider the linear VIE

$$t^\gamma u(t) = g(t) + \int_0^t \frac{\sqrt{3}}{3\pi}(t-s)^{-\mu} s^{\gamma+\mu-1} e^t u(s) ds, \quad t \in [0,1].$$

where $g(t)$ is a given function such that the exact solution of this problem is $u(t) = (t^{a_1} + t^{a_2})e^{-t}$.

- Set $a_1 = 1/3$, $a_2 = \sqrt{2}$, $\gamma = 2/3$ and $\mu = 2/3$. The exact solution u has a weak singularity at $t = 0$.
- Take $\lambda = 1/3$. In this case, $u(t^{\frac{1}{\lambda}})$ is not infinitely smooth.

Table 5: The convergence of fractional collocation for Example 2.

M	$\lambda = 1/3\ (p = 3, q = 4)$					
	$N = 1$		$N = 2$		$N = 3$	
	e_N	r	e_N	r	e_N	r
64	3.20E-04	1.92	4.65E-07	3.00	4.78E-09	3.83
128	8.33E-05	1.94	5.79E-08	3.01	3.20E-10	3.90
256	2.14E-05	1.96	7.17E-09	3.01	2.10E-11	3.93
EOC		2		3		4

Table 6: The convergence of polynomial collocation for Example 2.

M	$\lambda = 1\ (p = 3, q = 4)$					
	$N = 1$		$N = 2$		$N = 3$	
	e_N	r	e_N	r	e_N	r
64	2.11E-03	1.00	6.31E-04	1.00	3.56E-04	1.00
128	1.05E-04	1.00	3.15E-04	1.00	1.78E-04	1.00
256	5.27E-04	1.00	1.58E-04	1.00	8.89E-05	1.00
EOC		1		1		1

Overview

Summary

- A fractional collocation method was proposed for third-kind Volterra integral equations with non-compact kernel and non-smooth solution.
- The solvability and error analysis were studied based on a modified graded mesh.
- For solutions with initial weak singularity, the optimal convergence order can be achieved.

Thank you for your attention!