On discontinuous and continuous approximations to second-kind Volterra integral equations

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Outline

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Volterra integral equations (VIEs)

A linear VIE of the second kind on $t \in [0, T]$ is a functional equation of the form:

$$(V2): \quad u(t) = g(t) + \int_0^t K(t,s)u(s)\,ds,$$

$$(V2)_\alpha: \quad u(t) = g(t) + \int_0^t (t-s)^{-\alpha} K(t,s)u(s)\,ds, \;\; 0 < \alpha < 1.$$

Here, $g(t)$ and $K(t,s)$ are given functions, and $u(t)$ is an unknown function. The function $K(t,s)$ is called the kernel of the VIE.

A linear VIE of the first kind on $t \in [0, T]$ is given by

$$(V1): \quad \int_0^t K(t,s)u(s)\,ds = g(t),$$

$$(V1)_\alpha: \quad \int_0^t (t-s)^{-\alpha} K(t,s)u(s)\,ds = g(t), \;\; 0 < \alpha < 1.$$

Here, the unknown function occurs only under the integral sign.

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Applications

- Population dynamics, spread of epidemics
- Identification of memory kernel in viscoelasticity and heat conduction
- Retarded potential equations
- ...

Example: For the time evolution of the temperature u at the surface of a conducting solid where there is a high thermal loss:

$$u(t) = \pi^{-1/2} \int_0^t [f(s) - \gamma u^n(s)]ds, \ t \geq 0.$$

Here, γ – the ratio of the radiative properties to the conductive properties of the solid material $n = 4$–Stefan's radiation law $n = 1$–Newton's law of cooling

H. Brunner, Collocation Methods for Volterra Integral and Related Functional Differential Equations, Cambridge University Press, Cambridge, 2004a.

H. Brunner, Volterra Integral Equations: An Introduction to Theory and Applications, Cambridge University Press, Cambridge, 2017.

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Fractional differential equations (FDEs) and VIEs

Consider the following Caputo FDE

$$D_*^\alpha u(t) = f(t, u(t)), 0 < \alpha < 1$$

with initial value $u(0) = u_0$, where

$$D_*^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds.$$

The above fractional initial value problem can be transformed into the following VIE

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds.$$

K. Diethelm, The analysis of fractional differential equations, Lecture Notes in Mathematics, An application-oriented exposition using differential operators of Caputo type, Springer-Verlag, Berlin, 2010.

Outline

Recall

$$(V2): \quad u(t) = g(t) + \int_0^t K(t,s)u(s)\,ds.$$

Theorem (Existence and uniqueness, Brunner, Monograph, 2004)

Let $K \in C(D)$ and R denote the resolvent kernel associated with K. Then for any $g \in C(I)$, (V2) has a unique solution $u \in C(I)$, and this solution is given by

$$u(t) = g(t) + \int_0^t R(t,s)g(s)ds, \ t \in I.$$

Theorem (Regularity, Brunner, Monograph, 2004)

Assume that $K \in C^m(D)$. Then $R \in C^m(D)$. Thus, for any $g \in C^m(I)$ the solution of (V2) satisfies $u \in C^m(I)$.

The regularity of VIEs

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Assume that $K \in C(D)$, and let $0 < \alpha < 1$. Then for any $g \in C(I)$, $(V2)_\alpha$ possesses a unique solution $u \in C(I)$.

Theorem (Regularity, Brunner, Monograph, 2004)

Assume that $g \in C^m(I)$ and $K \in C^m(D)$, with $K(t,t) \neq 0$ on I. Then:

(i) For any $\alpha \in (0,1)$, the regularity of the unique solution of $(V2)_\alpha$ is described by

$$u \in C^m(0,T] \cap C(I), \;\text{with } |u'(t)| \le C_\alpha t^{-\alpha} \text{ for } t \in (0,T].$$

(ii) The solution u can be written in the form

$$u(t) = \sum_{(j,k)_\alpha} \gamma_{j,k}(\alpha) t^{j+k(1-\alpha)} + Y_m(t;\alpha), \;\; t \in I.$$

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Numerical methods

Discontinuous approximations in $S_{m-1}^{(-1)}(I_h)$

—the nature approximate space for VIEs

- Discontinuous collocation (DC) methods for (V2), $(V2)_\alpha$ and (V1): Brunner, Monograph, (2004) $(V1)_\alpha$ Conjecture
- Discontinuous Galerkin (DG) methods for (V2): Zhang, Lin & Rao, Appl. Math., (2000),
- DG methods for (V1): Brunner, Davies & Duncan, IMANA, (2009)
- Brunner, Davies & Duncan, IMANA, (2012): Established a relationship between quadrature DG (QDG) methods and DC methods for (V1).

Question: How about DG methods for $(V2)_\alpha$ and $(V1)_\alpha$?

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Continuous approximations in $S_m^{(0)}(I_h)$

- Continuous collocation (CC) methods for (V1): Kauthen & Brunner, MC, (1997)

—better convergence order

Question 1: How about CC methods for (V2), $(V2)_\alpha$ and $(V1)_\alpha$?

Question 2: How about CG methods for (V2), $(V2)_\alpha$, (V1) and $(V1)_\alpha$?

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DG methods for (V2)

Review: Discontinuous piecewise polynomial collocation (DC) methods

- Meshes: Let

$$I_h := \{t_n := nh : n = 0, 1, \ldots, N \ (t_N := T)\}$$

be a given mesh on $I = [0, T]$, with mesh diameter $h := T/N$.

- Discontinuous piecewise polynomials space:

$$S_{m-1}^{(-1)}(I_h) := \{v : \ v|_{\sigma_n} \in \mathcal{P}_{m-1} \ (0 \leq n \leq N-1)\},$$

where $\mathcal{P}_k = \mathcal{P}_k(\sigma_n)$ is the linear space of (real) polynomials of degree not exceeding k at the interval $\sigma_n := (t_n, t_{n+1}]$.

- Collocation points: For prescribed collocation parameters $\{c_i\}$, the set of collocation points is

$$X_h := \{t_n + c_i h_n : \ 0 < c_1 < \cdots < c_m \leq 1 \ \ (0 \leq n \leq N-1)\}.$$

The collocation equation is

$$u_{DC}(t) = g(t) + \int_0^t K(t,s)u_{DC}(s)\,ds, \qquad t \in X_h,$$

with the local representation of the DC solution

$$u_{DC}(t_n + sh) = \sum_{j=1}^{m} L_j(s)\,(U_{DC}^n)_j\,, \text{ with } L_j(s) := \prod_{k=1,k\neq j}^{m} \frac{s-c_k}{c_j-c_k},$$

where $s \in (0,1]$ and $(U_{DC}^n)_j := u_{DC}(t_{n,j})$.

Denote

$$\mathbf{G}_{DC}^n := \left(g(t_{n,1}),\ldots,g(t_{n,m})\right)^T, \qquad U_{DC}^n := \left((U_{DC}^n)_1\,,\ldots,(U_{DC}^n)_m\right)^T,$$

$$\mathbf{B}_{DC}^n := \begin{pmatrix} \int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(s)\,ds \\ (i,j=1,\ldots,m) \end{pmatrix}, \quad \mathbf{B}_{DC}^{(n,l)} := \begin{pmatrix} \int_0^{1} K(t_{n,i}, t_l + sh)L_j(s)\,ds \\ (i,j=1,\ldots,m) \end{pmatrix}.$$

Then the collocation equation can be written as

$$\left(\mathbf{I}_m - h\mathbf{B}_{DC}^n\right)\mathbf{U}_{DC}^n = \mathbf{G}_{DC}^n + h\sum_{l=0}^{n-1} \mathbf{B}_{DC}^{(n,l)}\mathbf{U}_{DC}^l.$$

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Fully discretised discontinuous collocation (FDC) methods

In general, the integrals of $B_{DC}^{n}, B_{DC}^{(n,l)}$ cannot be found analytically, but have to be approximated by suitable numerical quadrature formulas.

On σ_n, we choose interpolatory m-point quadrature formulas whose abscissas are based on the m collocation parameters $\{c_i\}$, and denote $b_i := \int_0^1 L_i(s)\,ds$ as the corresponding weights. Then

$$\int_0^{c_i} K(t_{n,i}, t_n + sh)L_j(s)\,ds \approx c_i \sum_{k=1}^{m} K(t_{n,i}, t_n + c_i c_k h)L_j(c_i c_k)b_k,$$

$$\int_0^{1} K(t_{n,i}, t_l + sh)L_j(s)\,ds \approx \sum_{k=1}^{m} K(t_{n,i}, t_n + c_k h)L_j(c_k)b_k = K(t_{n,i}, t_{l,j})b_j.$$

Therefore,

$$\left(\mathbf{I}_m - h\hat{\mathbf{B}}_{DC}^{n}\right)\hat{\mathbf{U}}_{DC}^{n} = \mathbf{G}_{DC}^{n} + h\sum_{l=0}^{n-1}\hat{\mathbf{B}}_{DC}^{(n,l)}\hat{\mathbf{U}}_{DC}^{l}.$$

DG methods

We are looking for the DG solution $u_{DG} \in S_{m-1}^{(-1)}(I_h)$, such that for $\forall \phi \in S_{m-1}^{(-1)}(I_h)$,

$$\int_{t_n}^{t_{n+1}} u_{DG}(s)\phi(s)\,ds = \int_{t_n}^{t_{n+1}} g(s)\phi(s)\,ds + \int_{t_n}^{t_{n+1}} \left(\int_0^s K(s,v)u_{DG}(v)\,dv\right)\phi(s)\,ds.$$

The local representation of the DG solution on the subinterval σ_n:

$$u_{DG}(t_n + sh) = \sum_{j=0}^{m-1} P_j(s)\,(U_{DG}^n)_j\,, \ \ s \in (0,1],$$

where $P_j(s)$ $(j = 0, \ldots, m-1)$ denote the 'shifted' Legendre polynomials of degree j on $[0,1]$, and $(U_{DG}^n)_j$ are unknown coefficients to be determined. Denote

$$\mathbf{G}_{DG}^n := \left(\int_0^1 g(t_n + sh_n)P_i(s)\,ds \ \ (i = 0, \ldots, m-1)\right)^T,$$

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$$\mathbf{B}_{DG}^{n} := \begin{pmatrix} \int_0^1 \left(\int_0^s K(t_n + sh, t_l + vh) P_j(v)\, dv\right) P_i(s)\, ds \\ (i, j = 0, \ldots, m-1) \end{pmatrix},$$

$$\mathbf{B}_{DG}^{(n,l)} := \begin{pmatrix} \int_0^1 \left(\int_0^1 K(t_n + sh, t_l + vh) P_j(v)\, dv\right) P_i(s)\, ds \\ (i, j = 0, \ldots, m-1) \end{pmatrix}.$$

Then

$$\left(\mathbf{A}_{DG} - h\mathbf{B}_{DG}^{n}\right) \mathbf{U}_{DG}^{n} = \mathbf{G}_{DG}^{n} + h \sum_{l=0}^{n-1} \mathbf{B}_{DG}^{(n,l)} \mathbf{U}_{DG}^{l}.$$

Note that

$$\mathbf{A}_{DG} := \begin{pmatrix} \int_0^1 P_i(s) P_j(s)\, ds \\ (i, j = 0, \ldots, m-1) \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \frac{1}{3} & & \\ & & \ddots & \\ & & & \frac{1}{2m-1} \end{pmatrix}$$

is nonsingular. For sufficiently small h, there determines a unique DG solution.

Remark

If a different set of basis functions is employed, the resulting DG solutions are equivalent.

QDG schemes and the relationship with DC schemes

QDG schemes is obtained from DG schemes: approximating the inner product by suitable numerical quadrature formulas.

On σ_n, suppose that the quadrature nodes and weights are based on $\{d_i\}_{i=1}^q$ and $\{w_i\}_{i=1}^q$, respectively, where $q \geq m$, $0 \leq d_1 < \cdots < d_q \leq 1$, and at least m weights are nonzero. Employing the basis functions $L_j(s)$, we have

$$\sum_{j=1}^{m}\sum_{k=1}^{q} L_j(d_k)L_i(d_k)w_k\left(\bar{U}_{DG}^n\right)_j = \sum_{k=1}^{q} g(t_n + d_k h)L_i(d_k)w_k$$

$$+ \sum_{k=1}^{q}\Bigg[h\sum_{j=1}^{m}\int_0^{d_k} K(t_n + d_k h, t_n + vh)L_j(v)\,dv\left(\bar{U}_{DG}^n\right)_j$$

$$+ \sum_{l=0}^{n-1} h\sum_{j=1}^{m}\int_0^1 K(t_n + d_k h, t_l + vh)L_j(v)\,dv\left(\bar{U}_{DG}^l\right)_j\Bigg]L_i(d_k)w_k.$$

Now consider the special case with $q = m$ and $d_k = c_k$. Then

$$\sum_{j=1}^{m}\sum_{k=1}^{m} L_j(c_k)L_i(c_k)w_k \left(\bar{U}_{DG}^n\right)_j = \sum_{k=1}^{m} g(t_n + c_k h)L_i(c_k)w_k$$

$$+ \sum_{k=1}^{m}\Bigg[h\sum_{j=1}^{m}\int_0^{c_k} K(t_n + c_k h, t_n + vh)L_j(v)\,dv\,\left(\bar{U}_{DG}^n\right)_j$$

$$+ \sum_{l=0}^{n-1} h\sum_{j=1}^{m}\int_0^{1} K(t_n + c_k h, t_l + vh)L_j(v)\,dv\,\left(\bar{U}_{DG}^l\right)_j\Bigg]L_i(c_k)w_k.$$

For $w_i \neq 0$, we obtain

$$\left(\bar{U}_{DG}^n\right)_i = g(t_{n,i}) + h\sum_{j=1}^{m}\int_0^{c_i} K(t_{n,i}, t_n + vh)L_j(v)\,dv\,\left(\bar{U}_{DG}^n\right)_j$$

$$+ \sum_{l=0}^{n-1} h\sum_{j=1}^{m}\int_0^{1} K(t_{n,i}, t_l + vh)L_j(v)\,dv\,\left(\bar{U}_{DG}^l\right)_j,$$

which is exactly the DC scheme.

Theorem (Liang, NMTMA, 2022)

Suppose that the inner products are approximated by m-point quadrature with nonzero weights $w_1, \ldots, w_m$ *and nodes* $0 < d_1 < \ldots < d_m \leq 1$. *Then the resulting QDG scheme is identical to the DC scheme with the collocation parameters* $\{c_i\}_{i=1}^m = \{d_i\}_{i=1}^m$ *whatever the choice of weights.*

FDG schemes and the relationship with FDC schemes

FDG schemes is obtained from QDG schemes: approximating the integral by suitable numerical quadrature formulas.

Similar to FDC, we obtain

$$\left(\hat{U}^n_{DG}\right)_i = g(t_{n,i}) + hc_i \sum_{j=1}^{m} \sum_{k=1}^{m} K(t_{n,i}, t_n + c_i c_k h) L_j(c_i c_k) b_k \left(\hat{U}^n_{DG}\right)_j$$
$$+ \sum_{l=0}^{n-1} h \sum_{j=1}^{m} K(t_{n,i}, t_{l,j}) b_j \left(\hat{U}^l_{DG}\right)_j,$$

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Theorem [Liang, NMTMA, 2022]

The resulting FDG scheme is identical to the FDC scheme.

Error analysis

Recall the error analysis of DC methods

Theorem (DC methods, Brunner, Monograph, 2004)

Assume that $g \in C^m(I), K \in C^m(D)$, *and* $u_h \in S_{m-1}^{(-1)}(I_h)$ *is the collocation solution for (V2). Then*

$$\|u - u_{DC}\|_\infty := \sup_{t \in I} |u(t) - u_{DC}(t)| \leq C\|u^{(m)}\|_\infty h^m.$$

holds for any set X_h *of collocation points with* $0 \leq c_1 < \cdots < c_m \leq 1$.

Theorem (FDC methods, Brunner, Monograph, 2004)

The FDC solution $\hat{u}_{DC}$ *has the same convergence property as* u_{DC}.

Error analysis for DG methods

Theorem (Zhang, Lin & Rao, 2000; Liang, NMTMA, 2022)

Assume:

(a) $g \in C^m(I)$ *and* $K \in C^m(D)$.

(b) u *and* $u_{DG} \in S_{m-1}^{(-1)}(I_h)$ *are the exact solution and the DG solution.*

Then for sufficiently small h,

$$\|u - u_{DG}\|_\infty := \sup_{t \in I} |u(t) - u_{DG}(t)| \leq C_{DG} \|u^{(m)}\|_\infty h^m.$$

Outline

DG methods for $(\text{V2})_\alpha$

Recall

$$(V2)_\alpha: \quad u(t) = g(t) + \int_0^t (t-s)^{-\alpha} K(t,s) u(s)\, ds, \quad 0 < \alpha < 1,$$

with regularity

$$u \in C^m(0,T] \cap C(I), \text{ with } |u'(t)| \le C_\alpha t^{-\alpha} \text{ for } t \in (0,T].$$

↪ Graded mesh: $I_h := \left\{ t_n := \left(\frac{n}{N}\right)^r T : \ n = 0, 1, \ldots, N \right\}$ with $N \ge 2$ and $r \ge 1$.

We are looking for the DG solution $u_h \in S_{m-1}^{(-1)}(I_h)$, such that for $\forall \phi \in S_{m-1}^{(-1)}(I_h)$,

$$\int_{t_n}^{t_{n+1}} u_h(s)\phi(s)\, ds = \int_{t_n}^{t_{n+1}} g(s)\phi(s)\, ds + \int_{t_n}^{t_{n+1}} \left(\int_0^s (s-v)^{-\alpha} K(s,v) u_h(v)\, dv \right) \phi(s)\, ds.$$

The local representation of the DG solution:

$$u_h(t_n + s h_n) = \sum_{j=0}^{m-1} P_j(s) U_{n,j}, \quad s \in (0,1],$$

where $\left(U_{DG}^n\right)_j$ are unknown coefficients to be determined.

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$$(V2)_\alpha: \quad u(t) = g(t) + \int_0^t (t-s)^{-\alpha} K(t,s) u(s)\, ds, \;\; 0 < \alpha < 1,$$

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$$u \in C^m(0,T] \cap C(I), \text{ with } |u'(t)| \le C_\alpha t^{-\alpha} \text{ for } t \in (0,T].$$

$\hookrightarrow$ Graded mesh: $I_h := \left\{ t_n := \left(\frac{n}{N}\right)^r T: \;\; n = 0, 1, \ldots, N \right\}$ with $N \ge 2$ and $r \ge 1$.

We are looking for the DG solution $u_h \in S_{m-1}^{(-1)}(I_h)$, such that for $\forall \phi \in S_{m-1}^{(-1)}(I_h)$,

$$\int_{t_n}^{t_{n+1}} u_h(s)\phi(s)\, ds = \int_{t_n}^{t_{n+1}} g(s)\phi(s)\, ds + \int_{t_n}^{t_{n+1}} \left(\int_0^s (s-v)^{-\alpha} K(s,v) u_h(v)\, dv \right) \phi(s)\, ds.$$

The local representation of the DG solution:

$$u_h(t_n + s h_n) = \sum_{j=0}^{m-1} P_j(s) U_{n,j}, \;\; s \in (0,1],$$

where $\left(U_{DG}^n\right)_j$ are unknown coefficients to be determined.

Denote

$$\mathbf{G}_n := \left(\int_0^1 g(t_n + sh_n) P_0(s)\, ds, \ldots, \int_0^1 g(t_n + sh_n) P_{m-1}(s)\, ds \right)^T,$$

$$\mathbf{U}_n := (U_{n,0}, \ldots, U_{n,m-1})^T, \mathbf{A} := \left(\begin{array}{c} \int_0^1 P_j(s) P_i(s)\, ds \\ (i,j = 0, \ldots, m-1) \end{array} \right),$$

$$\mathbf{B}_n(\alpha) := \left(\begin{array}{c} \int_0^1 \left(\int_0^s (s-v)^{-\alpha} K(t_n + sh_n, t_n + vh_n) P_j(v)\, dv \right) P_i(s)\, ds \\ (i,j = 0, \ldots, m-1) \end{array} \right),$$

$$\mathbf{B}^{(n,l)}(\alpha) := \left(\begin{array}{c} \int_0^1 \left(\int_0^1 \left(\frac{t_n + sh_n - t_l}{h_l} - v \right)^{-\alpha} K(t_n + sh_n, t_l + vh_l) P_j(v)\, dv \right) P_i(s)\, ds \\ (i,j = 0, \ldots, m-1) \end{array} \right).$$

Therefore,

$$\left(\mathbf{A} - h_n^{1-\alpha} \mathbf{B}_n(\alpha) \right) \mathbf{U}_n = \mathbf{G}_n + \sum_{l=0}^{n-1} h_l^{1-\alpha} \mathbf{B}^{(n,l)}(\alpha) \mathbf{U}_l.$$

For sufficiently small h, there determines a unique DG solution.

QDG methods for $(\mathrm{V2})_\alpha$

Theorem (Liang, ANM, 2022)

Suppose that the inner products are approximated by m-point quadrature with nonzero weights $w_1, \ldots, w_m$ and nodes $0 < d_1 < \ldots < d_m \leq 1$. Then the resulting QDG scheme is identical to the DC scheme with the collocation parameters $\{c_i\}_{i=1}^m = \{d_i\}_{i=1}^m$ whatever the choice of weights.

Error estimate for $(\mathrm{V2})_\alpha$

Theorem (Liang, ANM, 2022)

Assume:

(a) $g \in C^m(I)$ *and* $K \in C^m(D)$ *with* $K(t,t) \neq 0$.

(b) u *and* $u_{DG} \in S_{m-1}^{(-1)}(I_h)$ *are the exact solution and the DG solution.*

Then for sufficiently small h,

$$\|u - u_h\|_\infty := \sup_{t\in I} |u(t) - u_h(t)| \leq C\, h^{\min\{r(1-\alpha),\, m\}}.$$

Remark

The convergence order for the DG method in $S_{m-1}^{(-1)}(I_h)$ *is as same as the one for the DC method in the same polynomial space.*

Outline

Outline

Continuous methods for (V2)

CC methods

We seek the CC solution u_{CC} in the piecewise polynomial space

$$S_m^{(0)}(I_h) := \{v \in C(I): \ v|_{\bar{\sigma}_n} \in \mathcal{P}_m \ (0 \le n \le N-1)\}$$

of *continuous* piecewise polynomials of degree $m \ge 0$.

At $t = t_{n,i}$, the collocation equation reads as

$$u_{CC}(t) = g(t) + \int_0^t K(t,s) u_{CC}(s)\, ds, \qquad t \in X_h,$$

with $u_{CC}(0) = g(0)$. The local representation of the CC solution

$$u_{CC}(t_n + sh) = \sum_{j=0}^{m} l_j(s) \left(U_{CC}^n\right)_j, \ \ s \in [0,1],$$

where $\left(U_{CC}^n\right)_0 := u_{CC}(t_n)$, $\left(U_{CC}^n\right)_j := u_{CC}(t_{n,j})$ for $j = 1, \dots, m$, and

$$l_0(s) := \prod_{i=1}^{m} \frac{s - c_i}{-c_i}, \qquad l_j(s) := \frac{s}{c_j} \prod_{i=1, i \ne j}^{m} \frac{s - c_i}{c_j - c_i}.$$

Error analysis for CC methods

Theorem (CC methods, Liang & Brunner, BIT, 2016)

The CC solution $u_{CC} \in S_m^{(0)}(I_h)$ converges to the exact solution u if, and only if, the collocation parameters $\{c_i\}$ satisfy the condition

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \leq 1.$$

The corresponding attainable global order of convergence is given by

$$\max_{t\in I} |u(t) - u_h(t)| \leq C \begin{cases} h^{m+1}, & \text{if } -1 \leq \rho_m < 1, \\ h^m, & \text{if } \rho_m = 1. \end{cases}$$

Remark: The condition $-1 \leq \rho_m \leq 1$ is also the same sufficient and necessary condition to ensure the convergence of DC methods for $(V1)$.

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Remark: The condition $-1 \leq \rho_m \leq 1$ is also the same sufficient and necessary condition to ensure the convergence of DC methods for $(V1)$.

Error analysis for FCC methods

Theorem (FCC methods, Liang, NMTMA, 2022)

The FCC solution $\hat{u}_{CC}$ has the same convergence property as the CC solution u_{CC}.

Outline

CG methods for (V2)

We are looking for the CG solution $u_{CG} \in S_m^{(0)}(I_h)$ such that for $0 \leq n \leq N-1$ and any $\eta \in \mathcal{P}_{m-1}$,

$$\int_{t_n}^{t_{n+1}} u_{CG}(s)\eta(s)\,ds = \int_{t_n}^{t_{n+1}} g(s)\eta(s)\,ds + \int_{t_n}^{t_{n+1}} \int_0^s K(s,v)u_{CG}(v)\,dv\eta(s)\,ds.$$

Here, because of the continuity of $u_{CG}(t)$, we have $u_{CG}(t_{n-1}) = \lim_{t \to t_{n-1}^-} u_{CG}(t)$ $= \lim_{t \to t_{n-1}^+} u_{CG}(t)$. Hence $u_{CG}(t)$ has only m degrees of freedom on each subinterval, so $\eta \in \mathcal{P}_{m-1}$ (see Huang, Xu & Brunner (2016)).

The local representation of the CG solution

$$u_{CG}(t_n + sh) = \sum_{j=0}^{m} P_j(s)\left(U_{CG}^n\right)_j, \ \ s \in [0,1],$$

where the unknown coefficients $\left(U_{CG}^n\right)_j$ are to be determined.

CG methods for (V2)

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where the unknown coefficients $\left(U_{CG}^n\right)_j$ are to be determined.

Denote

$$\mathbf{U}_{CG}^n := \left((U_{CG}^n)_0, \ldots, (U_{CG}^n)_m\right)^T, \ \mathbf{A}_{CG}^n := \left(\begin{array}{c} \int_0^1 P_j(s)P_i(s)\,ds \\ (i = 0, \ldots, m-1; j = 0, \ldots, m) \end{array} \right),$$

$$\mathbf{B}_{CG}^n := \left(\begin{array}{c} \int_0^1 \left(\int_0^s K(t_n + sh, t_n + vh)P_j(v)\,dv\right) P_i(s)\,ds \\ (i = 0, \ldots, m-1; j = 0, \ldots, m) \end{array} \right),$$

$$\mathbf{B}_{CG}^{(n,l)} := \left(\begin{array}{c} \int_0^1 \left(\int_0^1 K(t_n + sh, t_l + vh)P_j(v)\,dv\right) P_i(s)\,ds \\ (i = 0, \ldots, m-1; j = 0, \ldots, m) \end{array} \right).$$

Then by the continuity,

$$\left(\begin{array}{c} (P_0(0), \ldots, P_m(0)) \\ (\mathbf{A}_{CG}^n - h\mathbf{B}_{CG}^n) \end{array} \right) \mathbf{U}_{CG}^n$$

$$= \left(\begin{array}{c} (P_0(1), \ldots, P_m(1)) \\ \mathbf{0}_{m\times(m+1)} \end{array} \right) \mathbf{U}_{CG}^{n-1} + h\sum_{l=0}^{n-1} \left(\begin{array}{c} \mathbf{0}_{1\times(m+1)} \\ \mathbf{B}_{CG}^{(n,l)} \end{array} \right) \mathbf{U}_{CG}^l + \left(\begin{array}{c} 0 \\ \mathbf{G}_{DG}^n \end{array} \right).$$

Note: Whatever the choice of basis functions, the resulting CG solutions are equivalent.

QCG schemes and the relationship with CC schemes

Theorem (Liang, NMTMA, 2022)

Suppose that the inner products are approximated by $m+1$-point quadrature with nonzero weights $w_0, \ldots, w_m$ and nodes $0 < d_0 < \ldots < d_m \leq 1$. Then the resulting QCG scheme is identical to the CC scheme with the collocation parameters $\{c_i\}_{i=0}^m = \{d_i\}_{i=0}^m$ whatever the choice of weights.

FCG schemes and the relationship with FCC schemes

Theorem (Liang, NMTMA, 2022)

The resulting FCG scheme is identical to the FCC scheme.

Error analysis for CG methods

Theorem (CG methods, Liang, NMTMA, 2022)

Assume:

(a) $g \in C^{m+2}(I)$ *and* $K \in C^{m+2}(D)$.

(b) u *and* $u_{CG} \in S_m^{(0)}(I_h)$ *are the exact solution and the CG solution.*

Then for sufficiently small h,

$$\|u - u_{CG}\|_\infty := \max_{t \in I} |u(t) - u_{CG}(t)| \le C_{CG} \begin{cases} h^{m+1}, & \text{if } m \text{ is odd;} \\ h^m, & \text{if } m \text{ is even.} \end{cases}$$

Remark: The convergence of the CG method for (V2) depends on the parity of m, which is similar to the convergence of the DG method for (V1).

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Outline

CC methods for $(V2)_\alpha$

Recall

$$(V2)_\alpha: \quad u(t) = g(t) + \int_0^t (t-s)^{-\alpha} K(t,s) u(s)\, ds, \;\; 0 < \alpha < 1,$$

with regularity

$$u \in C^m(0,T] \cap C(I), \text{ with } |u'(t)| \le C_\alpha t^{-\alpha} \text{ for } t \in (0,T].$$

$\hookrightarrow$ Graded mesh: $I_h := \left\{ t_n := \left(\frac{n}{N}\right)^r T : \; n = 0, 1, \ldots, N \right\}$ with $N \ge 2$ and $r \ge 1$.

We seek the CC solution u_{CC} in the piecewise polynomial space $S_m^{(0)}(I_h)$

At $t = t_{n,i}$, the collocation equation reads as

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CC methods for $(V2)_\alpha$

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Error analysis

Theorem (CC methods, Liang & Brunner, SINUM, 2019)

The CC solution $u_{CC} \in S_m^{(0)}(I_h)$ converges to the exact solution u if, and only if, the collocation parameters $\{c_i\}$ satisfy the condition

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \leq 1.$$

The corresponding attainable global order of convergence is given by

$$\max_{t\in I} |u(t) - u_h(t)| \leq C \begin{cases} h^{\min\{r(1-\alpha), m+1\}}, & \text{if } -1 \leq \rho_m < 1, \\ h^{\min\{r(1-\alpha), m\}}, & \text{if } \rho_m = 1. \end{cases}$$

Remark: The sufficient and necessary $-1 \leq \rho_m \leq 1$ does not depend on the weak singularity of $(\mathrm{V2})_\alpha$.

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Remark: The sufficient and necessary $-1 \leq \rho_m \leq 1$ does not depend on the weak singularity of $(\mathrm{V2})_\alpha$.

Outline

Numerical examples

Example

We take $K(t,s) \equiv 1$, $g(t) = 2e^{-t} - 1$. It is easy to check that the exact solution $u(t) = e^{-t}$.

表: The errors of DG solution for (V2) with $m = 1$

N	$\max_{1 \le n \le N} \lvert e(t_n) \rvert$	Order
2^5	1.5222e-02	-
2^6	7.7112e-03	0.98
2^7	3.8809e-03	0.99
2^8	1.9468e-03	1.00

表: The errors of DG solution for (V2) with $m = 2$

N	$\max_{1 \le n \le N} \lvert e(t_n) \rvert$	Order
2^5	7.9621e-05	-
2^6	2.0124e-05	1.98
2^7	5.0585e-06	1.99
2^8	1.2681e-06	2.00

表: The errors of DG solution for (V2) with $m = 3$

N	$\max_{1\le n\le N} \lvert e(t_n)\rvert$	Order
2^5	2.4926e-07	-
2^6	3.1472e-08	2.99
2^7	3.9537e-09	2.99
2^8	4.9546e-10	3.00

表: The errors of CG solution for (V2) with $m = 1$

N	$\max_{1\le n\le N} \lvert e(t_n)\rvert$	Order
2^5	1.5939e-04	-
2^6	4.0268e-05	1.98
2^7	1.0120e-05	1.99
2^8	2.5365e-06	2.00

表: The errors of CG solution for (V2) with $m = 2$

N	$\max_{1\leq n\leq N} \lvert e(t_n)\rvert$	Order
2^5	9.1726e-06	-
2^6	2.2932e-06	2.00
2^7	9.1726e-06	2.00
2^8	5.7329e-07	2.00

表: The errors of CG solution for (V2) with $m = 3$

N	$\max_{1\leq n\leq N} \lvert e(t_n)\rvert$	Order
2^5	1.1152e-09	-
2^6	7.0327e-11	3.99
2^7	4.4150e-12	3.99
2^8	2.7645e-13	4.00

Example

We take $K(t,s) = e^{t-s}$, $g(t) = \frac{3e^{-t}-e^{t}}{2}$. It is easy to check that it has the same exact solution as the above example.

表: The errors of FDC solution for (V2) with $m = 1$.

N	$c_1 = 0.1$ $(L_0(1) = -9)$	$c_1 = 0.49$ $(L_0(1) = -\frac{51}{49})$	$c_1 = 0.5$ $(L_0(1) = -1)$	$c_1 = 0.8$ $(L_0(1) = -\frac{1}{4})$	$c_1 = 1$ $(L_0(1) = 0)$
2^9	4.2859e-03	9.9420e-04	9.7466e-04	2.5996e-03	4.5758e-03
2^{10}	2.1473e-03	4.9757e-04	4.8780e-04	1.2996e-03	2.2867e-03
2^{11}	1.0748e-03	2.4890e-04	2.4402e-04	6.4978e-04	1.1431e-03
2^{12}	5.3766e-04	1.2448e-04	1.2204e-04	3.2488e-04	5.7146e-04
Order	1.00	1.00	1.00	1.00	1.00

表: The errors of FDC solution for (V2) with $m = 2$.

N	Gauss $(L_0(1) = 1)$	Radau IIA $(L_0(1) = 0)$	$(\frac{1}{4}, 1)$ $(L_0(1) = 0)$	$(\frac{1}{4}, \frac{5}{6})$ $(L_0(1) = \frac{3}{5})$	$(\frac{1}{6}, \frac{1}{2})$ $(L_0(1) = 5)$
2^5	7.9291e-05	3.9654e-06	1.8582e-04	5.9454e-05	4.5007e-04
2^6	2.0082e-05	4.9586e-07	4.7038e-05	1.5059e-05	1.1325e-04
2^7	5.0533e-06	6.1992e-08	1.1832e-05	3.7896e-06	2.8401e-05
2^8	1.2674e-06	7.7495e-09	2.9670e-06	9.5053e-07	7.1109e-06
Order	2.00	3.00	2.00	2.00	2.00

表: The errors of FDC solution for (V2) with $m = 3$.

N	Gauss $(L_0(1) = -1)$	Radau IIA $(L_0(1) = 0)$	$(\frac{1}{3}, \frac{1}{2}, 1)$ $(L_0(1) = 0)$	$(\frac{1}{3}, \frac{1}{2}, \frac{8}{9})$ $(L_0(1) = \frac{1}{4})$	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$ $(L_0(1) = 16)$
2^2	1.0804e-04	2.4056e-06	1.2600e-03	8.2916e-04	2.2833e-03
2^3	1.4823e-05	7.6075e-08	1.6374e-04	1.0643e-04	3.0343e-04
2^4	1.9414e-06	2.3845e-09	2.0828e-05	1.3442e-05	3.8873e-05
2^5	2.4843e-07	7.4574e-11	2.6248e-06	1.6876e-06	4.9128e-06
2^6	3.1419e-08	2.3323e-12	3.2939e-07	2.1138e-07	6.1730e-07
Order	2.98	5.00	2.99	3.00	2.99

表: The errors of FCC solution for (V2) with $m = 1$.

N	$c_1 = 0.1$ $(L_0(1) = -9)$	$c_1 = 0.49$ $(L_0(1) = -\frac{51}{49})$	$c_1 = 0.5$ $(L_0(1) = -1)$	$c_1 = 0.8$ $(L_0(1) = -\frac{1}{4})$	$c_1 = 1$ $(L_0(1) = 0)$
2^8	8.0094e+237	3.2351e-02	7.7556e-06	3.2485e-06	1.1904e-05
2^9	NaN	2.2706e+02	1.9458e-06	8.1121e-07	2.9760e-06
2^{10}	NaN	4.4652e+10	4.8730e-07	2.0269e-07	7.4399e-07
2^{11}	NaN	6.9019e+27	1.2193e-07	5.0657e-08	1.8600e-07
2^{12}	NaN	6.5934e+62	3.0496e-08	1.2662e-08	4.6499e-08
Order	-	-	2.00	2.00	2.00

表: The errors of FCC solution for (V2) with $m = 2$.

N	Gauss $(L_0(1) = 1)$	Radau IIA $(L_0(1) = 0)$	$(\frac{1}{4}, 1)$ $(L_0(1) = 0)$	$(\frac{1}{4}, \frac{5}{6})$ $(L_0(1) = \frac{3}{5})$	$(\frac{1}{6}, \frac{1}{2})$ $(L_0(1) = 5)$
2^4	5.6609e-05	2.0711e-05	3.1315e-05	8.5480e-06	4.2966e+05
2^5	1.4178e-05	2.6177e-06	3.9421e-06	1.2517e-06	8.5170e+15
2^6	3.5461e-06	3.2896e-07	4.9441e-07	1.7256e-07	2.5256e+37
2^7	8.8663e-07	4.1227e-08	6.1901e-08	2.2894e-08	1.7273e+81
2^8	2.2166e-07	5.1600e-09	7.7438e-09	2.9630e-09	6.3743e+169
Order	2.00	3.00	3.00	2.95	-

表: The errors of FCC solution for (V2) with $m = 3$.

N	Gauss ($L_0(1) = -1$)	Radau IIA ($L_0(1) = 0$)	$(\frac{1}{3}, \frac{1}{2}, 1)$ ($L_0(1) = 0$)	$(\frac{1}{3}, \frac{1}{2}, \frac{8}{9})$ ($L_0(1) = \frac{1}{4}$)	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$ ($L_0(1) = 16$)
2^2	6.9677e-06	1.0860e-06	4.6205e-05	9.8469e-06	8.4353e-02
2^3	4.7048e-07	3.5947e-08	3.0413e-06	6.2312e-07	4.8013e+02
2^4	3.0572e-08	1.1518e-09	1.9442e-07	3.9030e-08	1.4464e+11
2^5	1.9484e-09	3.6413e-11	1.2278e-08	2.4392e-09	1.7531e+29
2^6	1.2297e-10	1.1453e-12	7.7124e-10	1.5240e-10	3.8167e+66
Order	3.99	4.99	3.99	4.00	-

Example

Let $K(t,s) = \frac{1}{10\Gamma(1-\alpha)}$ and $g(t) = 1$ such that the exact solution $u(t) = E_{1-\alpha}(\frac{t^{1-\alpha}}{10})$.

表: The errors of DG solution for $(V2)_\alpha$ with $m = 1$ and $\alpha = 0.1$

N	uniform mesh	Order	graded mesh	Order
2^2	1.3406e-02	-	1.4297e-02	-
2^3	7.0274e-03	0.93	7.2205e-03	0.99
2^4	3.7215e-03	0.92	3.6279e-03	0.99
2^5	1.9817e-03	0.91	1.8184e-03	1.00
2^6	1.0583e-03	0.90	9.1028e-04	1.00
2^7	5.6612e-04	0.90	4.5541e-04	1.00
2^8	3.0308e-04	0.90	2.2777e-04	1.00

表: The errors of DG solution for $(V2)_\alpha$ with $m = 1$ and $\alpha = 0.3$

N	uniform mesh	Order	graded mesh	Order
2^2	1.3558e-02	-	1.5343e-02	-
2^3	8.1082e-03	0.74	7.8273e-03	0.97
2^4	4.9031e-03	0.73	3.9511e-03	0.99
2^5	2.9853e-03	0.72	1.9848e-03	0.99
2^6	1.8253e-03	0.71	9.9468e-04	1.00
2^7	1.1190e-03	0.71	4.9791e-04	1.00
2^8	6.8705e-04	0.70	2.4910e-04	1.00

表: The errors of DG solution for $(V2)_\alpha$ with $m = 1$ and $\alpha = 0.5$

N	uniform mesh	Order	graded mesh	Order
2^2	1.2342e-02	-	1.5751e-02	-
2^3	8.4399e-03	0.55	8.1742e-03	0.95
2^4	5.8289e-03	0.53	4.1586e-03	0.97
2^5	4.0537e-03	0.52	2.0968e-03	0.99
2^6	2.8330e-03	0.52	1.0528e-03	0.99
2^7	1.9867e-03	0.51	5.2747e-04	1.00
2^8	1.3966e-03	0.51	2.6401e-04	1.00

表: The errors of DG solution for $(V2)_\alpha$ with $m = 1$ and $\alpha = 0.7$

N	uniform mesh	Order	graded mesh	Order
2^2	9.1812e-03	-	1.4894e-02	-
2^3	7.2310e-03	0.34	8.0287e-03	0.89
2^4	5.7292e-03	0.34	4.1542e-03	0.95
2^5	4.5612e-03	0.33	2.1115e-03	0.98
2^6	3.6453e-03	0.32	1.0643e-03	0.99
2^7	2.9224e-03	0.32	5.3430e-04	0.99
2^8	2.3487e-03	0.32	2.6769e-04	1.00

表: The errors of DG solution for $(V2)_\alpha$ with $m = 1$ and $\alpha = 0.9$

N	uniform mesh	Order	graded mesh	Order
2^2	3.6123e-03	-	9.9107e-03	-
2^3	3.3226e-03	0.12	6.4984e-03	0.61
2^4	3.0593e-03	0.12	3.6663e-03	0.83
2^5	2.8196e-03	0.12	1.9381e-03	0.92
2^6	2.6010e-03	0.12	9.9525e-04	0.96
2^7	2.4013e-03	0.12	5.0417e-04	0.98
2^8	2.2185e-03	0.11	2.5373e-04	1.00

表: The errors of DG solution for $(V2)_\alpha$ with $m = 2$ and $\alpha = 0.1$

N	uniform mesh	Order	graded mesh	Order
2^2	7.8890e-05	-	1.6509e-04	-
2^3	5.6909e-05	0.47	4.8336e-05	1.77
2^4	3.4548e-05	0.72	1.3259e-05	1.87
2^5	1.9654e-05	0.81	3.4839e-06	1.93
2^6	1.0857e-05	0.86	8.9709e-07	1.96
2^7	5.9105e-06	0.88	2.2830e-07	1.97
2^8	3.1939e-06	0.89	5.7705e-08	1.98

表: The errors of DG solution for $(V2)_\alpha$ with $m = 2$ and $\alpha = 0.3$

N	uniform mesh	Order	graded mesh	Order
2^2	3.5720e-04	-	7.4088e-04	-
2^3	2.3223e-04	0.62	2.0797e-04	1.83
2^4	1.4734e-04	0.66	5.5664e-05	1.90
2^5	9.2299e-05	0.67	1.4461e-05	1.94
2^6	5.7407e-05	0.69	3.6995e-06	1.97
2^7	3.5559e-05	0.69	9.3788e-07	1.98
2^8	2.1972e-05	0.69	2.3650e-07	1.99

表: The errors of DG solution for $(V2)_\alpha$ with $m = 2$ and $\alpha = 0.5$

N	uniform mesh	Order	graded mesh	Order
2^2	5.8581e-04	-	1.8545e-03	-
2^3	4.1787e-04	0.49	5.2140e-04	1.83
2^4	2.9700e-04	0.49	1.3754e-04	1.92
2^5	2.1067e-04	0.50	3.5400e-05	1.96
2^6	1.4926e-04	0.50	8.9930e-06	1.98
2^7	1.0568e-04	0.50	2.2686e-06	1.99
2^8	7.4792e-05	0.50	5.7004e-07	1.99

表: The errors of DG solution for $(V2)_\alpha$ with $m = 3$ and $\alpha = 0.1$

N	uniform mesh	Order	graded mesh	Order
10	4.2362e-06	-	2.7848e-06	-
20	2.3002e-06	0.88	3.6816e-07	2.92
30	1.6047e-06	0.89	1.1139e-07	2.95
40	1.2419e-06	0.89	4.7514e-08	2.96
50	1.0176e-06	0.89	2.4498e-08	2.97
60	8.6451e-07	0.89	1.4246e-08	2.97

表: Uniform mesh: CC errors at the mesh points when $m = 2$ and $\alpha = 0.5$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, 1)$ ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^5	5.1326e-03	3.8744e-06	1.0609e-05	1.9691e-03	7.8034e+19
2^6	3.6490e-03	1.9313e-06	5.2714e-06	1.4007e-03	1.1272e+42
2^7	2.5901e-03	9.6354e-07	2.6241e-06	9.9457e-04	3.5959e+86
2^8	1.8372e-03	4.8102e-07	1.3079e-06	7.0535e-04	5.7691e+175
Order	0.50	1.00	1.00	0.50	-

表: Uniform mesh: CC errors at the mesh points when $m = 3$ and $\alpha = 0.5$.

N	Gauss ($\rho_m = -1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{3}, \frac{1}{2}, 1)$ ($\rho_m = 0$)	$(\frac{1}{3}, \frac{1}{2}, \frac{8}{9})$ ($\rho_m = \frac{1}{4}$)	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2})$ ($\rho_m = 16$)
2^4	5.1183e-03	1.0233e-06	9.6554e-06	4.0142e-04	7.2495e+16
2^5	3.6465e-03	5.0112e-07	4.7615e-06	2.8548e-04	8.6683e+35
2^6	2.5922e-03	2.4690e-07	2.3577e-06	2.0270e-04	1.8474e+74
2^7	1.8398e-03	1.2217e-07	1.1708e-06	1.4375e-04	1.2755e+151
Order	0.49	1.00	1.00	0.50	-

表: Graded mesh: CC errors at the mesh points when $m = 1$, $\alpha = 0.5$ and $r = 4$.

N	$c_1 = 0.1$ ($\rho_m = -9$)	$c_1 = 0.49$ ($\rho_m = -\frac{51}{49}$)	$c_1 = 0.5$ ($\rho_m = -1$)	$c_1 = 0.8$ ($\rho_m = -\frac{1}{4}$)	$c_1 = 1$ ($\rho_m = 0$)
2^8	3.2058e+237	9.3908e-04	9.2274e-07	5.4671e-07	6.8104e-08
2^9	-	4.7625e+00	2.3063e-07	1.3711e-07	1.7156e-08
2^{10}	-	5.9177e+08	5.7653e-08	3.4342e-08	4.3122e-09
2^{11}	-	4.7825e+25	1.4413e-08	8.5947e-09	1.0820e-09
Order	-	-	2.00	2.00	1.99

表: Graded mesh: CC errors at the mesh points when $m = 2$, $\alpha = 0.5$ and $r = 6$.

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{4}, 1)$ ($\rho_m = 0$)	$(\frac{1}{4}, \frac{5}{6})$ ($\rho_m = \frac{3}{5}$)	$(\frac{1}{6}, \frac{1}{2})$ ($\rho_m = 5$)
2^2	8.6626e-03	6.1892e-05	1.0288e-04	3.9837e-03	5.1577e-01
2^3	2.7930e-03	9.6685e-06	1.4758e-05	7.6029e-04	3.9099e+01
2^4	7.6970e-04	1.2945e-06	1.9123e-06	1.0729e-04	1.8153e+06
2^5	1.9476e-04	1.4389e-07	2.4403e-07	1.4011e-05	3.2306e+16
Order	1.98	3.17	2.97	2.94	-

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Conclusions and future work

Conclusions:

- Discontinuous methods
  - Convergence analysis of DG methods for (V2)
  - Convergence analysis of DG methods for $(V2)_\alpha$
- Discontinuous methods
  - Convergence analysis of CC methods for (V2)
  - Convergence analysis of CG methods for (V2)
  - Convergence analysis of CC methods for $(V2)_\alpha$

Future work: Convergence analysis of CG methods for $(V2)_\alpha$

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Welcome! *Thanks!*