Achieve higher efficiency at maximum power with finite-time quantum Otto cycle

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The optimization of heat engines was intensively explored to achieve higher efficiency while maintaining the output power. However, most investigations were limited to a few finite-time cycles, e.g., the Carnot-like cycle, due to the complexity of the finite-time thermodynamics. In this paper, we propose a class of finite-time engine with quantum Otto cycle, and demonstrate a higher achievable efficiency at maximum power. The current model can be widely utilized, benefitting from the general $C/\tau^2$ scaling of extra work for a finite-time adiabatic process with long control time $\tau$. We apply the adiabatic perturbation method to the quantum piston model and calculate the efficiency at maximum power, which is validated with an exact solution.

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I. INTRODUCTION

The emergent studies of quantum thermodynamics [1–6] have boosted the reminiscent investigation of heat engines into the microscopic level, especially on the optimizing performance [7–10] as well as the effect due to quantum coherence and correlations [11–15]. The key motivation is to optimize the heat engine by enhancing efficiency while maintaining the output power. Recently, significant effort has been devoted to optimizing the Carnot-like heat engine [9,10,16–19], similar to the Carnot cycle yet with finite operation time. The price to pay for such a finite-time cycle is the irreversible entropy production, which was found to be inversely proportional to the control time of the isothermal process. With this relation, the trade-off between efficiency and power is explicitly expressed via the constraint formula derived with different approaches [9,10,16,17,20–23], along with experimental attempts on the microscopic level [24–27].

Designing an optimal heat engine with a Carnot-like engine is a straightforward approach by noting what the Carnot bound is achieved by, yet should not be limited to. In theoretical investigation, quantum heat engines with finite-time Otto cycles hint at good performance [13,28,29], by utilizing phase transitions [28,30] or the specific control schemes [31–35]. However, the optimization of the finite-time quantum Otto heat engine remains vague, though with many pioneering investigations on concrete models [28,29,36–40], mainly due to the difficulty to include the effect of finite-time operations from a microscopic approach, especially that of the finite-time adiabatic processes. The evaluation of the finite-time effect of an adiabatic process is the key to the optimization of the quantum Otto cycle as well as the Carnot-like cycle.

In this paper, we overcome the current obstacle in optimizing the quantum Otto cycle by utilizing the high-order quantum adiabatic approximation [41–43]. In Sec. II, we show the universal $C/\tau^2$ scaling of the extra work during the adiabatic process with long control time $\tau$. The impact of the control scheme is reflected in the coefficient $C$ via nonadiabatic transitions between eigenstates. In Sec. III, we optimize the output power of the finite-time quantum Otto engine based on the $C/\tau^2$ scaling of the extra work. The efficiency at maximum power is found in an analytical form, which is probable to exceed that of the Carnot-like engine. In Sec. IV, the current formalism is applied to the piston model, which can be solved analytically to validate the $C/\tau^2$ scaling of the extra work. The conclusion is given in Sec. V.

II. $C/\tau^2$ SCALING OF EXTRA WORK IN THE FINITE-TIME ADIABATIC PROCESS

The quantum Otto cycle consists of two adiabatic and two isochoric processes. The work is performed in the two adiabatic processes, via changing the controllable parameters $\hat{R}(t)$, e.g., the volume for the trapped gas. The time for the isochoric process is typically negligible compared to that of the adiabatic process [44,45]. We thus focus on the finite-time quantum dynamics during the adiabatic process.

At the beginning of the adiabatic process, the system is initially prepared at a thermal equilibrium state

$$\rho(0) = \frac{e^{-\beta\hat{H}(0)}}{\text{Tr} e^{-\beta\hat{H}(0)}},$$

with inverse temperature $\beta$. $H[\hat{R}(t)]$ is the Hamiltonian with the control parameter $\hat{R}(t)$. The macroscopic parameters are tuned from $\hat{R}(0)$ at the beginning to $\hat{R}(\tau)$ at the end. The evolution of the system during $0 < t < \tau$ is controlled by a time-dependent Hamiltonian $H(t) = H[\hat{R}(t)]$ as

$$\dot{\rho} = -i[H(t), \rho].$$

Under the instantaneous basis $\{|n(t)\rangle\}$, the time-dependent Hamiltonian is diagonal,

$$H(t) = \sum_n E_n(t)|n(t)\rangle\langle n(t)|,$$
In the adiabatic process, the density matrix at any time in interval $[0, \tau]$ is
\[
\rho(t) = \sum_n p_n |\psi_n(t)\rangle \langle \psi_n(t)|,
\]
where $|\psi_n(t)\rangle$ follows the Schrödinger equation
\[
i\hbar \frac{\partial}{\partial t} |\psi_n(t)\rangle = H(t)|\psi_n(t)\rangle,
\]
with the initial state $|\psi_n(0)\rangle = |n(0)\rangle$. The evolution of the instantaneous state $|n(t)\rangle$ and the state $|\psi_n(t)\rangle$ is illustrated in Fig. 1(a). The purple-solid lines show the trajectories for finite-time adiabatic processes with changing control time $\tau$, and the black-dashed line presents the evolution of the instantaneous basis. With the increasing control time $\tau$, the state $|\psi_n(t)\rangle$ approaches the adiabatic trajectory $|n(t)\rangle$.

The finite-time effect is reflected through the work extraction during the adiabatic process. Since the system is isolated from any baths in the adiabatic processes, no heat is generated during the whole process. The work done is equal to the change of the internal energy $W(\tau) = \text{Tr}[\rho(\tau)H(\tau) - \rho(0)H(0)]$, explicitly as
\[
W(\tau) = \sum_n p_n [\langle \psi_n(\tau) | H(\tau) | \psi_n(\tau) \rangle - E_n(0)].
\]
To show the difference between the finite-time adiabatic process and its quasistatic counterpart, we define the extra work as
\[
W^{(\text{ex})}(\tau) = W(\tau) - W_{\text{adi}},
\]
where $W_{\text{adi}} = \sum_n p_n [E_n(\tau) - E_n(0)]$ is the work done during the quasistatic adiabatic process. One property of the extra work for the finite-time adiabatic process is its non-negativity $W^{(\text{ex})}(\tau) \geq 0$, which is proved with details in Appendix A. The non-negativity of the extra work ensures a lower efficiency of the finite-time quantum Otto cycle than that of the quasistatic one. Such non-negativity was previously known as the minimal work principle: for an initial thermal state, the quasistatic adiabatic process generates the minimal work when the energy level does not cross $|46\rangle$.

The key is to obtain the extra work via the dynamics of wave function $|\psi_n(t)\rangle$, which is expanded in the instantaneous basis $\{|l(t)\rangle\}$ as
\[
|\psi_n(t)\rangle = \sum_l c_{nl}(t) e^{-i\phi_l(t)} |l(t)\rangle,
\]
with a time-dependent phase. In turn, our definition of the extra work is appropriate in the sense of retaining the quantum adiabatic limit $|47\rangle$. Here, we clarify the distinction and connection between quantum adiabaticity and thermodynamic adiabaticity. Quantum adiabaticity means that population of the energy eigenstates remains unchanged during the whole process, while thermodynamic adiabaticity indicates no heat exchange between the system and the environment. In the finite-time adiabatic processes, the unitary evolution of an isolated quantum system ensures the thermodynamic adiabaticity, but quantum adiabaticity is not usually satisfied due to the nonadiabatic transition between different eigenstates. The rigorous quantum adiabaticity only holds at the infinite control time limit $\nu \to 0$. The quantum nonadiabaticity is responsible...
for the extra work needed to complete the adiabatic process in finite time.

With the amplitude $c_{nl}(\tau)$, the extra work by Eq. (8) is simplified as

$$W^{(\text{ex})}(\tau) = \sum_{n,l \neq n} p_n [\mathcal{E}_l(1) - \mathcal{E}_n(1)] |c_{nl}(\tau)|^2. \quad (13)$$

From the first-order adiabatic approximation result by Eq. (10), the value of the absolute square $|c_{nl}(\tau)|^2$, $n \neq l$ is divided into the mean part and the oscillating part as

$$|c_{nl}(\tau)|^2 = \left( |c_{nl}^{(\text{mean})}(\tau)|^2 \right)^{\text{(mean)}} + \left( |c_{nl}^{(\text{osc})}(\tau)|^2 \right)^{\text{(osc)}}, \quad (14)$$

where the mean part is

$$\left( |c_{nl}^{(\text{mean})}(\tau)|^2 \right)^{\text{(mean)}} = \frac{1}{\tau^2} (|\mathcal{T}_{nl}(1)|^2 + |\mathcal{T}_{nl}(0)|^2) \quad (15)$$

and the oscillating part is

$$\left( |c_{nl}^{(\text{osc})}(\tau)|^2 \right)^{\text{(osc)}} = -\frac{2}{\tau^2} \text{Re}\left[ e^{-i(\mathcal{E}_n(1) - \mathcal{E}_l(1)) + i[\phi_n(1) - \phi_l(1)]} \mathcal{T}_{nl}(1)^\dagger \mathcal{T}_{nl}(0) \right]. \quad (16)$$

To the first order, $|c_{nl}(\tau)|^2$ is proportional to $\nu^2$, leading to the $\mathcal{C}/\tau^2$ scaling of the extra work. Corresponding to Eqs. (15) and (16), the extra work $W^{(\text{ex})}(\tau) = W^{(\text{mean})}(\tau) + W^{(\text{osc})}(\tau)$ by Eq. (13) is divided into the mean extra work

$$W^{(\text{mean})}(\tau) = \sum_{\tau^2} \quad (17)$$

and the oscillating extra work

$$W^{(\text{osc})}(\tau) = \frac{\omega(\tau)}{\tau^2}. \quad (18)$$

The mean extra work decreases monotonously with the increasing control time $\tau$, while the oscillating extra work oscillates around zero, and contributes the fluctuation in the extra work. The coefficients in Eqs. (17) and (18) follow explicitly as

$$\Sigma = \sum_{n,l \neq n} p_n [\mathcal{E}_l(1) - \mathcal{E}_n(1)] [|\mathcal{T}_{nl}(1)|^2 + |\mathcal{T}_{nl}(0)|^2] \quad (19)$$

and

$$\omega(\tau) = -\sum_{n,l \neq n} 2p_n [\mathcal{E}_l(1) - \mathcal{E}_n(1)] \text{Re}\left[ \mathcal{T}_{nl}(1)^\dagger \mathcal{T}_{nl}(0) \right] \times e^{-i[\phi_n(1) - \phi_l(1)] + i[\phi_n(1) - \phi_l(1)]}. \quad (20)$$

The impact of the control scheme is reflected through the transition amplitude $\mathcal{T}_{nl}(s)$. Interestingly, the mean extra work, to the leading order, only depends on the initial (final) transition amplitude $\mathcal{T}_{nl}(0)$ [$\mathcal{T}_{nl}(1)$], instead of the whole trajectory. And the oscillating one relies on the trajectory only through the dynamical phase $\phi_n(s)$ and the Berry phase $\phi_n(s)$.

For the oscillating extra work, $\omega(\tau)$ oscillates around zero with the increasing control time $\tau$. When we consider the system with the incommensurable energy difference $\mathcal{E}_l(1) - \mathcal{E}_n(1)$ for different sets of indexes $l$ and $n$, the oscillation of $\omega(\tau)$ contains different frequency $\phi_n(1) - \phi_l(1)$. In the summation of $\omega(\tau)$, the terms with different phase $\phi_n(1) - \phi_l(1)$ cancel out each other. In the following discussion, we will neglect the oscillating term in Eq. (18). Yet, this oscillating term may introduce higher efficiency for a system with few energy levels, e.g., the two-level system [29,48].

The finite-time effect of the adiabatic process has been previously considered in the studies of the two-level quantum heat engines [37–39], by assuming a phenomenological dissipation inverse proportional to the control time. Yet, our strict derivation shows that the extra work follows the $\mathcal{C}/\tau^2$ scaling instead. In Ref. [48], we apply the current method and theorem to the two-level quantum Otto engine, and consider the contribution of both the mean and the oscillated extra work. The oscillation property of the extra work might be utilized to improve the performance of the finite-time quantum Otto engine.

III. EFFICIENCY AT MAXIMUM POWER FOR QUANTUM OTTO ENGINE

With the $\mathcal{C}/\tau^2$ scaling of the extra work, we evaluate the performance of the quantum Otto engine by the efficiency and the output power. The quantum Otto cycle is illustrated via the $(H) - \mathcal{R}$ diagram in Fig. 1(b). The solid line shows the finite-time quantum Otto cycle, while the dashed line shows the corresponding quasistatic one. The work done during the two adiabatic processes $(1 \rightarrow 2)$ and $(3 \rightarrow 4)$ are $W_1(\tau_1) < 0$ and $W_3(\tau_3) > 0$ with the change of external parameters ($R_0 \leftrightarrow R_1$), respectively. The heat engine contacts with the hot (cold) bath and reaches the equilibrium with the temperature $T_1$ ($T_3$) in the isochoric heating $(4 \rightarrow 1)$ [isochoric cooling $(2 \rightarrow 3)$] with the fixed parameter $R_0(R_1)$.

The performance of the quantum Otto engine is evaluated by the efficiency and the output power. We need the net work and the heat exchange under the adiabatic perturbation approximation. For the two adiabatic processes, the work is

$$W_i(\tau_i) = \text{Tr}[\rho_i(1) H_{i+1} - \text{Tr}[\rho_i H_i]] \quad (21)$$

$$= W_i^{\text{adi}} + \sum_{\tau_i} \xi_i \tau_i, \quad i = 1, 3, \quad (22)$$

where $\tau_i$ is the corresponding control time, and $\Sigma_i$ is the corresponding coefficient related to the control scheme. The work in the quasistatic adiabatic process is given by

$$W_i^{\text{adi}} = \text{Tr}[\rho_i^{\text{adi}} H_{i+1}] - \text{Tr}[\rho_i H_i] \quad (23)$$

We consider the relaxation time in the isochoric processes is much shorter than the control time $\tau_i$, $i = 1, 3$ in the adiabatic processes. The time consuming of the isochoric processes is neglected, and the system is fully thermalized after the isochoric processes. The heat exchange with the hot bath during the isochoric process is

$$Q_h = \text{Tr}[\rho_3 H_4] - \text{Tr}[\rho_1 H_1] \quad (24)$$

$$= Q_h^{\text{adi}} - \frac{\Sigma_3}{\tau_3} \quad (25)$$

The net work for the whole cycle is

$$W_T = -[W_1(\tau_1) + W_3(\tau_3)] \quad (26)$$

and the efficiency is

$$\eta = \frac{W_T}{Q_h}. \quad (27)$$
Combining the equations for $W_i(\tau_i)$ and $W_{\text{adi}}^i$, the power $P = W/(\tau_1 + \tau_3)$ for the finite-time quantum Otto engine follows explicitly,

$$P = \frac{W_{\text{adi}}^i}{\tau_1 + \tau_3} - \frac{1}{\tau_1 + \tau_3} \left( \frac{\Sigma_1}{\tau_1^2} + \frac{\Sigma_3}{\tau_3^2} \right),$$  \hspace{1cm} (28)

with the efficiency

$$\eta = \frac{W_{\text{adi}}^i - (\Sigma_1/\tau_1^2 + \Sigma_3/\tau_3^2)}{W_{\text{adi}}^i / \eta_{\text{adi}} - \Sigma_3/\tau_3^2}. \hspace{1cm} (29)$$

Here, $W_{\text{adi}}^i = -(W_{\text{adi}}^1 + W_{\text{adi}}^3)$ is the net work for the quasistatic Otto cycle with the corresponding efficiency $\eta_{\text{adi}} = W_{\text{adi}}^i / Q_{\text{adi}}^i$.

According to the optimal condition of the maximum power $\partial P / \partial \tau_1 = 0, \partial P / \partial \tau_3 = 0$, the current finite-time quantum Otto engine reaches its maximum power

$$P_{\text{max}} = 2 \left[ \frac{W_{\text{adi}}^i}{3(\Sigma_1^{1/3} + \Sigma_3^{1/3})} \right]^2, \hspace{1cm} (30)$$

at the optimal operation time $\tau_1^* = (3(\Sigma_1^{1/3} - \Sigma_3^{1/3}))/W_{\text{adi}}^{1/2}$ and $\tau_3^* = (3(\Sigma_3^{1/3} + \Sigma_3)/W_{\text{adi}}^{1/2})$. The corresponding efficiency at the maximum power (EMP) is

$$\eta_{\text{EMP}} = \frac{2\eta_{\text{adi}}}{3 - \eta_{\text{adi}}}, \hspace{1cm} (31)$$

which depends on the ratio $\Sigma_1/\Sigma_3$ and the efficiency $\eta_{\text{adi}}$ of the quasistatic Otto cycle. In the limit $\Sigma_1/\Sigma_3 \to 0$, the EMP reaches the upper bound

$$\eta_{\text{EMP}}^+ = \frac{2\eta_{\text{adi}}}{3 - \eta_{\text{adi}}}. \hspace{1cm} (32)$$

In the limit $\Sigma_1/\Sigma_3 \to \infty$, the EMP reaches the lower bound

$$\eta_{\text{EMP}}^- = \frac{2\eta_{\text{adi}}}{3 - \eta_{\text{adi}}}. \hspace{1cm} (33)$$

We obtain the main result in Eq. (31) with the first-order quantum adiabatic approximation, where the inverse control time $\nu$ is the perturbation parameter. The result relies on two key factors, i.e., the long control time $[41,43] \tau$ and the nonlevel crossing condition [46]. To obtain the EMP, we have neglected the oscillating extra work with the observation of incommensurability of the typical energy levels. Yet, such an oscillating part can introduce interesting effects on EMP for small quantum systems, e.g., the minimal quantum heat engine with a two-level system [48].

We turn to comparing the EMP of the finite-time quantum Otto cycle with that of the Carnot-like cycle. For the Carnot-like cycle, the upper (lower) bound $\eta_{\text{CL}}^+ = \eta_{\text{C}}/(2 - \eta_{\text{C}})$ ($\eta_{\text{CL}}^- = \eta_{\text{C}}/2$) is approached when the entropy production of the hot (cold) isothermal process dominates [9]. Here, $\eta_{\text{C}} = 1 - T_1/T_3$ is the Carnot efficiency for a heat engine working between the low temperature $T_1$ and the high temperature $T_3$ baths. The entropy production in the Carnot-like cycle is partly determined by the thermal conductivity of the heat bath, and can be modulated by tuning the interaction strength between the system and the bath. In the finite-time Otto cycle, the coefficients $\Sigma_1$ and $\Sigma_3$ are determined by the protocol of the adiabatic process and cannot be flexibly modulated. To allow a fair comparison, we set the highest (lowest) temperature $T_1$ ($T_3$) in the isochoric process to be the temperature for the hot (cold) bath, namely $T_1 = T_1$ ($T_3 = T_3$). To surpass the EMP of the Carnot-like heat engine ($\eta_{\text{CL}}^+ > \eta_{\text{CL}}^-$), it is required that

$$\eta_{\text{adi}} > \frac{3\eta_{\text{C}}}{4 - \eta_{\text{C}}}. \hspace{1cm} (34)$$

The efficiency $\eta_{\text{adi}}$ of the quasistatic Otto is always smaller than the Carnot efficiency $\eta_{\text{C}}$, namely, $\eta_{\text{adi}} < \eta_{\text{C}}$.

In Fig. 2, we show the EMP of both the Carnot-like cycle and the current quantum Otto cycle. We set the efficiency of the corresponding quasistatic Otto engine as $\eta_{\text{adi}} = \theta \eta_{\text{C}}$ with the ratio $\theta \in [0, 1]$. For the finite-time quantum Otto cycle, the upper (lower) bound $\eta_{\text{EMP}}^+ (\eta_{\text{EMP}}^-)$ is plotted as the red-solid (blue-dashed) line. Two sets of the ratios $\theta = 0.5$ and $\theta = 1$ are plotted. For the Carnot-like heat engine, the black-dash-dotted and the black-dotted lines give the upper bound $\eta_{\text{CL}}^+$ and the lower bound $\eta_{\text{CL}}^-$, respectively [9]. For $\theta = 1$, the curve shows that the EMP of the finite-time quantum Otto cycle exceeds the one for the finite-time Carnot cycle. Such a higher EMP is achievable only at the region $\theta > 3/4$. The curves for $\theta = 0.5$ show a lower efficiency than that of the Carnot-like cycle. The current generic model implies the possibility to surpass the EMP of the Carnot-like cycle by choosing the proper efficiency of the quasistatic Otto cycle $\eta_{\text{adi}}$ larger than $3\eta_{\text{C}}/4$. We will realize such a quantum Otto cycle with an example of the quantum piston model.

IV. FINITE-TIME QUANTUM OTTO ENGINE ON PISTON MODEL

We illustrate the efficiency of the quasistatic Otto cycle with a concrete model of a single particle trapped in a...
derive by the adiabatic perturbation theory. Here, the square box with the Hamiltonian,

\[ H(t) = -\frac{1}{2M} \dot{q}^2 + V(x, t), \]

(35)

where \( M \) is the mass of the particle and \( V(x, t) \) is the square potential

\[ V(x, t) = \begin{cases} \infty, & x < 0, x > L(t), \\ 0, & 0 \leq x \leq L(t). \end{cases} \]

(36)

The controllable length \( L(t) \) serves as the tuning parameter \( R(t) \) as discussed in the generic model. The advantage of the current model is the existence of an exact solution for the linear control protocol \([49, 50]\),

\[ L(t) = L_0 + (L_1 - L_0) \frac{t}{\tau}, \]

(37)

which allows a direct validation of the scaling in Eq. (17) derived by the adiabatic perturbation theory. Here, \( L_0 \) (\( L_1 \)) is the initial (final) length of the box during the adiabatic process.

For this control scheme \( \dot{L}(s) = L(\pi s) \), the instantaneous wave function is

\[ \langle x | \hat{n}(s) \rangle = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi}{L} x \right), \]

(38)

with the corresponding energy

\[ E_n(s) = \frac{n^2\pi^2}{2ML^2}. \]

(39)

The nonadiabatic transition rate is

\[ \tilde{T}_n(s) = -\frac{4Mn(1-n^2)(L_1 - L_0)}{\pi^2(n^2 - L_0^2)^2} \dot{L}(s). \]

(40)

Under the long control time limit, we obtain the asymptotic result of the extra work as

\[ W^{(\text{mean})}(\tau) = \frac{ML^2(1-r^2)(1 + r^2)}{\tau^2} \left( 1 - \sum_{n=1}^{\infty} \frac{p_n}{4\pi^2 n^2} \right), \]

(41)

with the expansion ratio \( r = L_0/L_1 \). The initial thermal distribution is

\[ p_n(\beta, L_0) = \frac{e^{-\frac{n^2\pi^2}{L^2} \beta}}{Z(\beta, L_0)}, \]

(42)

with the initial inverse temperature \( \beta = 1/k_BT \). The partition function is

\[ Z(\beta, L) = \frac{1}{2} [\partial_3(0, e^{-\frac{\beta^2\pi^2}{L^2} Q}) - \frac{1}{2}], \]

(43)

where \( \partial_3(0, q) = 2 \sum_{n=1}^{\infty} q^{-2} q^2 + 1 \) is the elliptic-theta function. The detailed derivation of Eq. (41) is given in Appendix C, where the oscillating extra work is also obtained analytically. At the high temperature limit \( \beta \to 0 \), the thermal de Broglie wavelength \( \lambda_B = (2\pi \beta/M)^{1/2} \) is much smaller than the length of the box and the summation \( \sum_{n=1}^{\infty} p_n/(4\pi^2 n^2) \) in Eq. (41) can be neglected. Thus we obtain the approximation for the mean extra work in Eq. (41) as

\[ W^{(\text{mean})}(\tau) \approx \frac{ML^2(1-r^2)(1 + r^2)}{6\tau^2}. \]

(44)

By controlling the length of the trap, we realize the finite-time quantum Otto cycle with the current quantum piston model. The two lengths for the adiabatic process are \( L_0 \) and \( L_1 \) with \( L_0 < L_1 \). For the quasistatic Otto cycle, the efficiency of the engine is \( \eta_{\text{adi}} = 1 - r^2 \) and the net work of the whole cycle has a simple result at high temperature:

\[ W^{\text{adi}} = \frac{k_B}{2} (T_s r^2 - T_c) \left( \frac{1}{r^2} - 1 \right). \]

(45)

For the finite-time adiabatic process, the coefficients of the mean extra work with linear control schemes are

\[ \Sigma_1 = \frac{M(1 - r^2)(1 + r^2)L_1^2}{6}, \]

(46)

and

\[ \Sigma_3 = \frac{M(1 - r^2)(1 + r^2)L_1^2}{6r^2}. \]

(47)

Figure 3(a) validates the \( C/r^2 \) scaling of the extra work \( W^{(\text{mean})}(\tau) \) in Eq. (44) during the expansion of the quantum piston model. We set the mass \( M = 1 \) and the Boltzmann constant as \( k_B = 1 \) in all the later calculations. During the expansion, the length of the box varies from the initial value \( L_0 = 1 \) to the final value \( L_1 = 2 \). Exact results are obtained with analytical solution of the time-dependent Schrödinger equation in Refs. [49–51]. We choose the initial thermal states with different temperatures \( T = 1, 50, \) and \( 100 \), marked with blue circle, black square, and red diamond, respectively. The oscillation of the extra work becomes weaker for higher temperature. For long control time, the exact numerical result of the extra work (the markers) matches with the analytical one (the green-solid line), demonstrating the \( C/r^2 \) scaling of the extra work.

The maximum power for the piston model is obtained as

\[ P_{\text{piston}}^{\text{max}} = \frac{1}{3L_1} \left( \frac{k_B(T_s r^2 - T_c)(1 - r^2)}{(M(1 - r^2)(1 + r^2))^2 (r^2 + r^2)} \right)^{1/2}, \]

(48)

by choosing the optimal control time \( \tau_1^* \) and \( \tau_3^* \). And the corresponding efficiency is obtained by Eq. (31)

\[ \eta_{\text{EMP}} = \frac{2\eta_{\text{adi}}}{3 - \eta_{\text{adi}}}. \]

(49)

The detailed derivation of Eqs. (48) and (49) together with the optimal control times \( \tau_1^* \) and \( \tau_3^* \) are given in Appendix C.

In Fig. 3(b), we plot the EMP \( \eta_{\text{EMP}}^{\text{on}} \) of the quantum piston model (the red-solid curve) as the function of the expansion ratio \( r \) with the general bound \( \eta_{\text{EMP}} \) (the green-dotted curve). The requirement of the positive power in Eq. (48) implies the constraint for the expansion ratio \( r \approx \sqrt{T_s/T_0} < r < 1 \), shown as the white area. The upper bound \( \eta_{\text{CL}} \) of the Carnot-like cycle is plotted as the horizontal black-dashed line, with the fixed temperature ratio \( T_s/T_0 = 1/2 \). Figure 3(b) indicates the higher EMP of the finite-time quantum Otto cycle than that of the Carnot-like cycle at the region \( 1/\sqrt{2} < r < 0.736 \), illustrated as the blue area. The number 0.736 is obtained by
FIG. 3. (a) $C/\tau^2$ scaling for the extra work in the finite-time adiabatic expansion process on the quantum piston model, with the length chosen as $L_0 = 1$ and $L_1 = 2$. The blue circle, black square, and red diamond show the exact numerical result with the initial temperatures $T = 1, 50, 100$, respectively. The green-solid line presents the analytical result of the mean extra work for large $\tau$ in Eq. (17). (b) The EMP $\eta_{\text{EMP}}^{\text{piston}}$ of the quantum piston model (the red-solid curve) as the function of the expansion ratio $r = L_0/L_1$, with the fixed temperature ratio $T_c/T_h = 1/2$. The green-dotted curve presents the general bound $\eta_{\text{EMP}}$ by Eq. (32). The black-dashed horizontal line presents the upper bound of the Carnot-like cycle $\eta_{\text{CL}}$. The gray area shows the engine with negative output power $P < 0$ and the blue area shows the higher EMP than that of the Carnot-like cycle.

solving the equation $\eta_{\text{piston}} = \eta_{\text{CL}}^{\text{piston}}$. In the current model, the piston is controlled with the simplest scheme that the expansion ratio is the only optimizing parameter for the EMP $\eta_{\text{EMP}}$. A more complicated control scheme [49] can be considered to show more flexible tuning EMP beyond the Carnot-like cycle.

In Figs. 4(a) and 4(b), we show the efficiency and the power for the finite-time quantum Otto cycle built on the quantum piston model. The temperatures of the hot bath and the cold bath are chosen as $T_h = 100, T_c = 20$. We optimize the control time $\tau_1, \tau_3$ of the finite-time adiabatic process to obtain the maximum power. Figure 4(a) shows that the power reaches maximum at the particular control time $\tau_1^* = 0.33, \tau_3^* = 0.52$ (the blue arrow). The adiabatic perturbation theory works well at these timescales according to Fig. 3(a). The corresponding EMP in Fig. 4(b) is given with the blue arrow. In Fig. 4(c), we show the constraint between the power and the efficiency, by randomly choosing 600,000 pairs of $(\tau_1, \tau_3)$ to calculate the corresponding power and efficiency. A clear bound appears, which shows a trade-off between the power and the efficiency. The maximum power along with the EMP is marked with the blue arrow. The numerical results of the maximum power and the EMP match with Eqs. (48) and Eq. (49), respectively.

V. CONCLUSION AND REMARKS

In this paper, we have studied the finite-time effect of adiabatic processes. With the high-order quantum adiabatic approximation, we have proved the universal $C/\tau^2$ scaling for the extra work in the finite-time adiabatic processes, and validated it with the quantum piston model. It is meaningful to test this universal scaling on other complex quantum systems from both theoretical and experimental aspects. The current experimental setup on the trapped Fermi gas [35,52] can be directly applied to verify the $C/\tau^2$ scaling of the extra work. One needs to choose a fixed protocol of the adiabatic process and measure the work to complete the adiabatic process for different control time $\tau$.

Moreover, we described a class of finite-time quantum heat engine with quantum Otto cycle. Instead of the irreversible entropy production in the Carnot-like cycles, the finite-time effect of the quantum Otto cycle is ascribed to the extra work. Importantly, we showed such a cycle is capable of achieving higher efficiency at maximum power than that of the widely used Carnot-like cycle. The better performance of the quantum Otto cycle will attract attention for new designs of the quantum heat engine, instead of focusing on optimizing the Carnot-like cycle in finite time. It is proposed that the quantum Otto cycle can be implemented on a single-ion.
engine [36,40]. Our study contributes to the further optimization of the concrete finite-time quantum engine in the experiments.

In the derivation of the $C/\tau^2$ scaling for the extra work, the oscillating extra work is neglected due to the incommensurable energy difference of the complex system. Yet, for the quantum heat engine with work matter consisting of few energy levels, the oscillating extra work can affect the performance of the quantum Otto engine. The oscillating behavior of the extra work leads to a quantum Otto engine with high efficiency [53,54], and will induce a different effect on the efficiency-power constraint relation [48].

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APPENDIX A: POSITIVITY OF THE EXTRA WORK

In this Appendix, we prove the positivity of the extra work by using the Schur-Horn theorem. We remark that the proof of the positive extra work was already presented elsewhere [46]. However, our version of the proof with the Schur-Horn theorem is different and straightforward. Generally, we assume the energy levels shift, retaining the order $E_1(s) < E_2(s) < \cdots < E_n(s) < \cdots$ during the whole adiabatic process [46]. For an initial thermal state, the extra work from Eq. (B1) is explicitly written as

$$W^{(ex)}(\tau) = \sum_{l=1}^{\infty} \lambda_{ll} E_{l}(1) - \sum_{n=1}^{\infty} p_n E_n(1), \quad (A1)$$

with the notation $\lambda_{ll} = \sum_{n=1}^{\infty} p_n |c_n(t)|^2$. We rearrange the summation and obtain

$$W^{(ex)}(\tau) = E_1(1) \left[ \sum_{l=1}^{\infty} (\lambda_{ll} - p_l) \right] + \sum_{j=2}^{\infty} (E_j(1) - E_{j-1}(1)) \times \left[ \sum_{l=j}^{\infty} (\lambda_{ll} - p_l) \right]. \quad (A2)$$

The first term on the right-hand side of Eq. (A2) is zero due to the normalized condition for the probability $\sum_{l=1}^{\infty} \lambda_{ll} = \sum_{l=1}^{\infty} p_l = 1$. We prove the second term on the right-hand side is non-negative based on the Schur-Horn theorem [55]. Since $E_j(1) - E_{j-1}(1) > 0$, we only need to prove $\sum_{l=j}^{\infty} \lambda_{ll} \geq \sum_{l=j}^{\infty} p_l$.

Here, $\lambda_{ll}$ can be regarded as the diagonal element for the Hermite matrix $\lambda_{im} = \sum_{p=1}^{\infty} p_n |c_n(t)|^2 e_{nm}(t)$, which is obtained from the diagonal matrix with the diagonal element $p_n$ through the unitary transform $e_{nm}(t)$. The eigenvalue of this Hermite matrix is exactly $p_n$. We resequence the diagonal terms $\lambda_{ll}$ in the nonincreasing order as $\tilde{\lambda}_{11} \geq \tilde{\lambda}_{22} \geq \cdots \geq \tilde{\lambda}_{nn} \cdots$. The Schur-Horn theorem [55] presents the following inequality:

$$\sum_{j=1}^{j-1} \tilde{\lambda}_{jj} \leq \sum_{j=1}^{j-1} p_l, \quad j \geq 2 \quad (A3)$$

for a Hermitian matrix with the diagonal terms $\tilde{\lambda}_{ll}$ and eigenvalue $p_l$ both in nonincreasing order. Together with the normalization of the probability, we have the inequality $\sum_{l=1}^{\infty} \tilde{\lambda}_{ll} \geq \sum_{l=1}^{\infty} p_l$, $j \geq 1$. Since $\tilde{\lambda}_{11}$ gives the nonincreasing order for $\lambda_{ll}$, we have apparently $\sum_{l=1}^{\infty} \tilde{\lambda}_{ll} \geq \sum_{l=1}^{\infty} \lambda_{ll}$, and thus $\sum_{l=1}^{\infty} (\tilde{\lambda}_{ll} - p_l) \geq 0$.

Therefore, we have proven $W^{(ex)}(\tau) \geq 0$: the extra work for an initial thermal state is non-negative when the energy level does not cross during the finite-time adiabatic process.

APPENDIX B: FIRST-ORDER ADIABATIC APPROXIMATION AND THE EXTRA WORK

This Appendix is devoted to showing the detailed derivation of the nonadiabatic correction for the extra work for finite-time adiabatic processes based on the higher-order adiabatic approximation [41]. The Schrödinger equation $i\partial_t |\psi(t)\rangle = H(t)|\psi(t)\rangle$ results in the following differential equation for the amplitude $c_{al}(t)$:

$$\frac{d}{dt} c_{al}(t) + c_{al}(t) \Gamma_{il}(t) + \sum_{m \neq l} c_{am}(t) e^{-i[\phi_m(t) - \phi_l(t)]} \Gamma_{im}(t) = 0,$$

with the dynamical phase $\phi_l(t) = \int_0^t E_l(t') dt'$ and the notation $\Gamma_{im}(t) = \langle l(t)|d/dt|m(t)\rangle$. We consider for a given protocol of the adiabatic process $\tilde{H}(s) = H(\tau s) = \sum_n E_n(s) |\tilde{n}(s)\rangle\langle \tilde{n}(s)|$, with $\tau$ as the control time, where $|\tilde{n}(s)\rangle = |n(t(s))\rangle$ and $E_(s) = E_n(s)$. Representing the amplitude $b_{al}(s) = c_{al}(t(s))$ with the rescaled time parameter $s$, the differential equation is rewritten for $b_{al}(s)$ as

$$\frac{d}{ds} b_{al}(s) + b_{al}(s) \tilde{\Gamma}_{il}(s) + \sum_{m \neq l} b_{am}(s) e^{-i[\phi_m(s) - \phi_l(s)]} \tilde{\Gamma}_{im}(s) = 0,$$

where the notation $\tilde{\Gamma}_{im}(s) = \langle \tilde{l}(s)|d/ds|\tilde{n}(s)\rangle$ and the dynamical phase $\phi_l(s) = \int_0^s E_l(s') ds'$ are given by the rescaled time parameter $s$.

Based on the high-order quantum adiabatic approximation in Ref. [41], we obtain the solution of $b_{al}^{(0)}(s) = b_{al}^{(0)}(s) + b_{al}^{(1)}(s)/\tau$ to the first order of $1/\tau$, where $b_{al}^{(0)}(s)$ and $b_{al}^{(1)}(s)$ satisfy the following differential equations:

$$\frac{d}{ds} b_{al}^{(0)}(s) + \tilde{\Gamma}_{il}(s) b_{al}^{(0)}(s) = 0, \quad (B3)$$

$$\frac{d}{ds} b_{al}^{(1)}(s) + \tilde{\Gamma}_{il}(s) b_{al}^{(1)}(s) + \sum_{m \neq l} \frac{d}{ds} (i\tilde{\Gamma}_{ml}(s) e^{-i[\phi_m(s) - \phi_l(s)]} b_{ml}^{(0)}(s)) = 0. \quad (B4)$$

Here, $\tilde{\Gamma}_{ml}(s) = \tilde{\Gamma}_{im}(s)/[E_m(s) - E_l(s)]$ denotes the nonadiabatic transition rate between the states $|\tilde{l}(s)\rangle$ and $|\tilde{n}(s)\rangle$. 062140-7
According to the initial condition $c_{nl}(0) = \delta_{ln}$, we attain the initial conditions $b_{nl}^{(0)}(0) = \delta_{ln}$ and $b_{nl}^{(1)}(0) = 0$ for Eqs. (B3) and (B4), respectively. The solutions to Eqs. (B3) and (B4) follow as

$$
\begin{align*}
\tilde{b}_{nl}^{(0)}(s) = \begin{cases} 
0, & n \neq l, \\
e^{i\tilde{\gamma}_l(s)}, & n = l,
\end{cases}
\tilde{b}_{nl}^{(1)}(s) = \begin{cases} 
-i[\tilde{T}_{nl}(s)e^{-iT[\tilde{\psi}_{nl}(s)]}] & n \neq l, \\
0, & n = l,
\end{cases}
\end{align*}
$$

(B5)

with the Berry phase $\tilde{\gamma}_l(s) = i \int_0^s \tilde{T} \cdot d\gamma'.$ In the main content, Eq. (3) is obtained via $c^{(1)}_{nl}(\tau) = b^{(1)}_{nl}(1)$. We remark that the current derivation of the adiabatic approximation is the straightforward version. A more careful derivation can be found in Ref. [43], where the first-order result for $c_{nl}^{(1)}(\tau)$ contains a phase correction. Yet such a phase has no effect on the absolute square $|c_{nl}^{(1)}(\tau)|^2$ and in turn would not change the results obtained from the current derivation.

For the initial thermal state, the work $W(\tau) = \sum_n p_n[\langle \psi_n(\tau) | H(\tau) | \psi_n(\tau) \rangle - \tilde{E}_n(0)]$ is given explicitly as

$$
W(\tau) = \sum_n p_n \left[ \tilde{E}_n(1) - \tilde{E}_n(0) + \sum_{1 \neq n} \left( \tilde{E}_n(1) - \tilde{E}_n(0) \right) |c_{nl}(\tau)|^2 \right].
$$

(B7)

Here, $p_n = \exp[-\beta \tilde{E}_n(0)] / \sum_n \exp[-\beta \tilde{E}_n(0)]$ denotes the initial thermal distribution with the inverse temperature $\beta = 1/(k_B T)$. For a quasistatic adiabatic process with long control time $\tau \rightarrow \infty$, the solution by Eq. (3) in the main content implies $|c_{nl}(\tau)|^2 \rightarrow 0$, $n \neq l$, and the corresponding work approaches $W_{\text{adi}} = \sum_n p_n[\tilde{E}_n(1) - \tilde{E}_n(0)]$. The rest part of the work in Eq. (B7) is named as the extra work for the finite-time adiabatic process

$$
W^{(\text{ex})}(\tau) = \sum_n p_n \left[ \sum_{1 \neq n} \left( \tilde{E}_n(1) - \tilde{E}_n(0) \right) |c_{nl}(\tau)|^2 \right],
$$

(B8)

which is Eq. (13) in the main content.

**APPENDIX C: 1D QUANTUM PISTON MODEL**

In this Appendix, we show the details about the realization of the finite-time quantum Otto cycle with a 1D quantum piston model. Explicit results of the maximal power and the EMP are derived for this model.

1. $C/\tau^2$ scaling of the extra work

First, we show the $C/\tau^2$ scaling of the extra work for a 1D quantum piston model during the finite-time adiabatic process. The time-dependent Hamiltonian $H(t)$ is given by Eq. (35) of the main content with the control protocol $L(s) = L_0 + (L_1 - L_0)s$. The instantaneous wave function and the corresponding energy $\tilde{E}_n(s)$ are given by Eqs. (38) and (39),

$$
\Gamma_{ln}(s) \text{ of this model follows explicitly as }
$$

$$
\tilde{\Gamma}_{ln}(s) = \begin{cases} 
0, & l \neq n,
\end{cases}
$$

(C1)

$$
\tilde{\Gamma}_{ln}(s) = \frac{2nL(-1)^{l+n}(L_1 - L_0)}{(l^2 - n^2\tilde{L}(s))}, & l \neq n.
$$

(C2)

Therefore, the Berry phase vanishes in this model, namely $\tilde{\gamma}_l = 0$. And the nonadiabatic transition rate is

$$
\tilde{T}_{nl}(s) = -\frac{4Mn\tilde{L}(1)}{\pi^2(n^2 - l^2)^2}L(s).
$$

(C3)

Substituting the rate into Eq. (3) in the main content, we obtain the amplitude explicitly as

$$
\tilde{c}_{nl}^{(1)}(\tau) = \frac{i}{\tau^{1/2}} \int_{1/2}^{1/2} \sum_{l \neq n} \tilde{E}_n(1) - \tilde{E}_n(0) \langle\psi_n(\tau)|e^{-iT|\psi_n(\tau)|}] \right).
$$

(C4)

By summing over the initial thermal distribution, we obtain the explicit result for the extra work,

$$
W^{(\text{ex})}(\tau) = W^{(\text{mean})}(\tau) + W^{(\text{osc})}(\tau),
$$

(C5)

where the mean extra work is

$$
W^{(\text{mean})}(\tau) = \frac{ML^2}{\tau^2} (1 - r)^2 (1 + r^2) \left( \frac{1}{6} - \sum_{n=1}^{\infty} \frac{p_n}{4\pi^2 n^2} \right)
$$

(C6)

and the oscillating extra work is

$$
W^{(\text{osc})}(\tau) = -\sum_{n=1}^{\infty} \frac{16ML^2}{\tau^2} (1 - r)^2 r \sum_{l \neq n} \frac{L^2 n^2}{\pi^2 (l^2 - n^2)^3} \times \cos \left( \frac{\tau (n^2 - l^2)^2}{2M L^2 n^2} \right)
$$

(C7)

where $r = L_0/L_1$ is the expansion ratio and $p_n = p_n(\beta, L_0)$ is the initial thermal distribution given by Eq. (42) in the main content. With Eqs. (C6) and (C7), the coefficients in Eq. (19) and Eq. (20) of the main content are written explicitly as

$$
\Sigma = ML^2 (1 - r)^2 (1 + r^2) \left( \frac{1}{6} - \sum_{n=1}^{\infty} \frac{p_n}{4\pi^2 n^2} \right)
$$

(C8)

and

$$
\omega(\tau) = -16ML^2 (1 - r)^2 r \sum_{n=1}^{\infty} \sum_{l \neq n} \frac{L^2 n^2}{\pi^2 (l^2 - n^2)^3} \times \cos \left( \frac{\tau (n^2 - l^2)^2}{2M L^2 n^2} \right).
$$

(C9)
For high temperature with the thermal de Broglie wavelength \( \lambda_{\text{th}} = \sqrt{2\pi \beta/M} \) much smaller than the length \( L_0 \) of the box, the summation by Eq. (C8) can be approximated as \( p_n/(4\pi^2 n^2) \approx j_{n+1/2}^2 / (4\pi^2 n^2 \pi) dn \). And the summation over the index \( n \) can be estimated as
\[
\sum_{n=1}^{\infty} p_n(\beta, L_0) 1/n^2 \approx \int_{1/2}^{\infty} 1/n^2 \exp\left(\frac{\beta x^2}{2ML^2}\right) \text{erfc}\left(\frac{\beta x}{\sqrt{8L^2 M}}\right) dn \quad \text{(C10)}
\]
\[
= \frac{e^{-\beta^2/8\pi L^2 M}}{\sqrt{\pi} \beta \sqrt{8\pi L^2 M}} - \frac{1}{2} \frac{\beta}{4L_0^2 M} \quad \text{(C11)}
\]
\[
= \frac{\sqrt{\beta}}{2\pi^2 L_0^2 M} + O(\beta). \quad \text{(C12)}
\]

Therefore, we neglect the last summation term in Eq. (C8) at the high temperature limit and simplify both coefficients \( \Sigma \) as
\[
\Sigma = \frac{M}{6} ML_0^2 (1 - r^2)(1 + r^2), \quad \text{(C13)}
\]
and the approximate mean extra work \( W(\text{mean}) \) is given by Eq. (44) in the main content.

\[
W(\tau) = \frac{1}{\tau^2} \sum_{n=1}^{\infty} p_n \left( \sum_{l=1}^{\infty} |\langle \Psi_l(0)|n(0)\rangle|^2 \langle \Psi_l(\tau)|H(\tau)|\Psi_l(\tau)\rangle - E_n(0) \right) \quad \text{(C17)}
\]

with \( H(\tau) \) given by Eq. (35) in the main content and the initial energy \( E_n(0) = n^2\pi^2/(2ML_0^2) \). The extra work follows from Eq. (B8) as
\[
W(\text{ex}) = \sum_{n=1}^{\infty} p_n \left( \sum_{l=1}^{\infty} |\langle \Psi_l(0)|n(0)\rangle|^2 \langle \Psi_l(\tau)|H(\tau)|\Psi_l(\tau)\rangle - E_n(0) \right). \quad \text{(C18)}
\]

The exact extra work is obtained by numerically calculating the initial projection \( \langle \Psi_l(0)|n(0)\rangle \) and the internal energy \( \langle \Psi_l(\tau)|H(\tau)|\Psi_l(\tau)\rangle \).

3. Validation of the scaling

With the exact solution, we can validate the obtained \( C/\tau^2 \) scaling. In addition to the expansion process [Fig. 3(a) in the main content], we supplement the \( C/\tau^2 \) scaling of the extra work in the compression process in Fig. 5(a). We set the mass and the Boltzmann constant as \( M = 1, k_B = 1 \), and consider three initial thermal equilibrium states with the temperature \( T = 1, 50, 100 \) [blue circle, black square, and red diamond in Fig. 5(a)]. For higher temperature, the oscillation of the extra work becomes weaker. And the exact numerical results match the mean extra work in Eq. (44) at high temperature (the green line). In Fig. 5(b), we compare the total extra work (the curves) in Eq. (C5) with the exact numerical results (the markers). The curves show a good match with the exact numerical results for both the compression and the expansion processes with long control time \( \tau \).

2. Exact solution

The current model can be solved analytically as shown in Refs. [49–51]. Here, we only show the relevant part of the exact solution for the later numerical calculations. For the given protocol above, the exact solution for the time-dependent Schrödinger equation \( i\partial_t|\Psi_\beta(t)\rangle = H(t)|\Psi_\beta(t)\rangle \) exists as
\[
\langle x|\Psi_\beta(t)\rangle = e^{i\left(\frac{1}{2}(\Delta_x^2 + \Delta_p^2 - \frac{\beta^2}{\pi t^2})\right)} \sqrt{\frac{2}{L(t)}} \sin \frac{n\pi x}{L(t)}, \quad \text{(C14)}
\]
with \( L(t) = L_0 + (L_1 - L_0)t/\tau \). Here, the time-dependent solution \( |\Psi_\beta(t)\rangle \) forms a complete orthogonal set at any given time \( t \). Therefore, the initial eigenstate \( |\psi_n(0)\rangle = |n(0)\rangle \) can be expanded with \( |\Psi_\beta(t)\rangle \) as
\[
|n(0)\rangle = \sum_{l=1}^{\infty} \langle \psi_l(0)|n(0)\rangle|\Psi_l(0)\rangle, \quad \text{(C15)}
\]
and the state at time \( \tau \) follows as
\[
|\psi_n(\tau)\rangle = \sum_{l=1}^{\infty} \langle \psi_l(0)|n(0)\rangle|\Psi_l(\tau)\rangle. \quad \text{(C16)}
\]

For an initial thermal state with the distribution \( p_n = p_n(\beta, L_0) \), the work is determined by the change of the internal energy

4. Engine cycle

To optimize a finite-time quantum Otto engine, we need the net work \( W_{\text{T}}^{\text{adi}} \) and the efficiency \( \eta_{\text{adi}} \) for the quasistatic Otto cycle. Considering the quantum Otto cycle given in Fig. 1(b) in the main content, the internal energy for the equilibrium states 1 and 3 is \( \text{Tr}[\rho_1 H_1] = \sum_{n=1}^{\infty} p_n^{(1)} \tilde{E}_n(0) \) and \( \text{Tr}[\rho_3 H_1] = \sum_{n=1}^{\infty} p_n^{(3)} \tilde{E}_n(1) \) with the thermal distribution \( p_n^{(1)} = p_n(\beta_1, L_0) \) and \( p_n^{(3)} = p_n(\beta_3, L_1) \). For a quasistatic Otto cycle, the distribution during the quasistatic adiabatic processes remains its initial distribution, which leads to the internal energy for the states 2 and 4 as \( \text{Tr}[\rho_2 H_2] = \sum_{n=1}^{\infty} p_n^{(2)} \tilde{E}_n(1) \) and \( \text{Tr}[\rho_4 H_1] = \sum_{n=1}^{\infty} p_n^{(3)} \tilde{E}_n(0) \). We obtain the heat absorbed from the hot bath as
\[
Q_{\text{h}}^{\text{adi}} = \sum_{n=1}^{\infty} (p_n^{(1)} - p_n^{(3)}) \tilde{E}_n(0) \quad \text{(C19)}
\]
and the net work as
\[
W_{\text{T}}^{\text{adi}} = \sum_{n=1}^{\infty} (p_n^{(1)} - p_n^{(3)}) (\tilde{E}_n(1) - \tilde{E}_n(0)). \quad \text{(C20)}
\]
the temperature expansion and compression process. The length of the box changes from $L_0$ to $L_1$. The exact numerical results for three initial thermal equilibrium states with temperatures $T = 1, 50, 100$ are presented in the blue circles, black squares, and red diamonds. And the green line shows the mean extra work $W^{(\text{mean})}$ in Eq. (44). (b) The extra work for the expansion and compression process. The length of the box changes from $L_0 = 2(1)$ to $L_1 = 1(2)$ for the compression (expansion) process at the temperature $T = 100$. The upper red (lower blue) line with markers presents the total extra work for the compression (expansion) process. The markers show the exact numerical results, while the line is obtained by Eq. (C5).

The efficiency for the quasistatic Otto cycle follows as

$$\eta^{\text{adi}} = 1 - r^2,$$  \hspace{1cm} (C21)

with the ratio $r = L_0/L_1$. At high temperature, the summation in Eq. (C20) can be approximated as

$$W^{\text{adi}}_T \approx \frac{k_B}{2} (T_h r^2 - T_c) \frac{1 - r^2}{r^2}.$$  \hspace{1cm} (C22)

For the current model with 1D quantum piston, the extra work at high temperature is determined by the coefficients

$$\Sigma_1 = \frac{ML^2}{6} (1 - r)^2 (1 + r^2)$$  \hspace{1cm} (C23)

and

$$\Sigma_3 = \frac{ML^2}{6r^2} (1 - r)^2 (1 + r^2).$$  \hspace{1cm} (C24)

With the explicit result of Eqs. (C21)–(C24), the optimal control time follows as

$$\tau^*_1 = \sqrt{\frac{ML^2}{k_B (T_h r^2 - T_c) (1 + r^2) r^4/3 + r^2}}.$$  \hspace{1cm} (C25)

According to Eqs. (30) and (31), the maximal power and the EMP of the quantum piston model are obtained as

$$p_{\text{max}}^{\text{piston}} = \frac{1}{3L_1} \left[ \frac{k_B (T_h r^2 - T_c) (1 - r^2)}{(M(1 - r^2)(1 + r^2))^{1/3}(r^2 + r^{2/3})} \right]^{1/2},$$  \hspace{1cm} (C27)

and

$$\eta_{\text{EMP}}^{\text{piston}} = \frac{2(1 - r^2)}{3 - (1 - r^2)/(1 + r^{2/3})},$$  \hspace{1cm} (C28)

respectively. The efficiency can be rewritten with the quasistatic efficiency $\eta^{\text{adi}}$ as

$$\eta_{\text{EMP}}^{\text{piston}} = \frac{2\eta^{\text{adi}}}{3 - \eta^{\text{adi}}[(1 - \eta^{\text{adi}})^{1/3} + 1]},$$  \hspace{1cm} (C29)

which is Eq. (49) in the main content.


